# Lagrangian Rabinowitz Floer homology and twisted cotangent bundles

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**Abstract**

We study the Lagrangian intersection theoretic version of Rabinowitz Floer homology, which we define for virtually contact $\pi_1$-injective hypersurfaces and virtually exact $\pi_1$-injective Lagrangians in symplectically aspherical geometrically bounded symplectic manifolds. By means of an Abbondandolo-Schwarz short exact sequence we then compute the Lagrangian Rabinowitz Floer homology of certain Mañé supercritical hypersurfaces in twisted cotangent bundles, where the Lagrangians are conormal bundles.

## 1 Introduction

In this paper we investigate the **Lagrangian Rabinowitz Floer homology** $\mathrm{RFH}_*^{\alpha}(\Sigma, L, X)$ associated to a hypersurface $\Sigma$ and a Lagrangian submanifold $L$, in a symplectic manifold $X$, all in the **virtually contact** setting. This is the Lagrangian intersection theoretic version of Rabinowitz Floer homology $\mathrm{RFH}_*^{\alpha}(\Sigma, X)$, which was introduced by Cieliebak and Frauenfelder in [18], and then extended by several other authors ([4, 10, 20, 19, 12, 13, 33, 9, 28, 29]). The starting point of Rabinowitz Floer homology is to work with a different action functional than the one normally used in Floer homology. This functional was originally introduced by Rabinowitz [38], and has the advantage that its critical points detect periodic orbits of the Hamiltonian lying on a **fixed energy level**. Thus Rabinowitz Floer homology is an invariant of a pair $(\Sigma, X)$, where $(X, \omega)$ is a symplectic manifold and $\Sigma \subseteq X$ is a hypersurface. Actually certain assumptions are required on both $\Sigma$ and $X$ in order for the Rabinowitz Floer homology $\mathrm{RFH}_*^{\alpha}(\Sigma, X)$ to be defined; for instance, one could assume that $\omega = d\lambda$ is exact, $X$ is convex at infinity and $\Sigma$ is a hypersurface of restricted contact type (this was the setting originally studied in [18]; since then Rabinowitz Floer homology has been extended to considerably more general situations). Its usefulness stems from the fact that if $\Sigma$ is displaceable in $X$ then $\mathrm{RFH}_*^{\alpha}(\Sigma, X) = 0$. We refer the reader to [11] for a detailed survey of the construction of Rabinowitz Floer homology, and for the applications this homology theory has generated so far.

Our first result summarizes the main properties of Lagrangian Rabinowitz Floer homology in the virtually contact setting. We begin by explaining in detail the various assumptions we make, together with the various other definitions needed in order to understand the statement of Theorem A.

**Assumptions on the ambient symplectic manifold:**

Let $(X^{2n}, \omega)$ denote a connected symplectic manifold.
We assume throughout that:

1. $(X, \omega)$ is **geometrically bounded** - this means that there exist $\omega$-compatible almost complex structures $J$ on with the property that the Riemannian metric $g_J(\cdot,\cdot) := \omega(J\cdot,\cdot)$ is complete, has bounded sectional curvature and has injectivity radius bounded away from zero.

2. The first Chern class $c_1(TX)$ is zero.

3. The symplectic form $\omega$ is **symplectically aspherical**. This means that for every smooth map $f : S^2 \to X$, one has $\int_{S^2} f^*\omega = 0$.

Assumption (2) is made for simplicity only, and could be weakened at the expense of losing the $\mathbb{Z}$-grading on the Lagrangian Rabinowitz Floer homology. Assumption (1) however is much more crucial, and cannot be dispensed with. If we denote by $\mathsf{p} : \widetilde{X} \to X$ the universal cover of $X$ and by $\widetilde{\omega} := \mathsf{p}^*\omega \in \Omega^2(\widetilde{X})$ then Assumption (3) is equivalent to requiring that $\widetilde{\omega}$ is exact.

**Assumptions on the hypersurface:**

A hypersurface $\Sigma \subseteq X$ is $\pi_1$**-injective** if the map $\iota_{\Sigma *} : \pi_1(\Sigma) \to \pi_1(X)$ induced by the inclusion $\iota_\Sigma : \Sigma \hookrightarrow X$ is injective. A closed connected orientable hypersurface $\Sigma \subseteq X$ is **separating** if $X \backslash \Sigma$ has two components, of which precisely one is relatively compact. Recall that for a closed connected orientable hypersurface $\Sigma \subseteq X$, there is a distinguished oriented line bundle $\mathfrak{L}_\Sigma \to \Sigma$ over $\Sigma$ called the **characteristic line bundle**. This is the line bundle over $\Sigma$ defined by

$$\mathfrak{L}_\Sigma(x) := \{ v \in T_x\Sigma \, : \, \omega_x(v, w) = 0 \text{ for all } w \in T_x\Sigma \} .$$

The characteristic line bundle determines a foliation $\mathfrak{F}_\Sigma$ of $\Sigma$. We write $\mathscr{D}(\Sigma) \subseteq C^\infty(X, \mathbb{R})$ for the set of **defining Hamiltonians** for $\Sigma$. By definition these are Hamiltonians $H : X \to \mathbb{R}$ such that 0 is a regular value of $H$, $\Sigma = H^{-1}(0)$, and such that the symplectic gradient $X_H$ of $H$ is a positively oriented section of $\mathfrak{L}_\Sigma$. If $H \in \mathscr{D}(\Sigma)$ then given $x \in \Sigma$ the leaf $\mathfrak{F}_\Sigma(x)$ of the foliation $\mathfrak{F}_\Sigma$ containing $x$ is simply the orbit of $x$ under the flow $\phi^H_t$ of $X_H$, that is,

$$\mathfrak{F}_\Sigma(x) = \phi^H_{\mathbb{R}}(x).$$

We write $\mathscr{D}_0(\Sigma)$ for the subset of Hamiltonians $H \in \mathscr{D}(\Sigma)$ with the additional property that $\mathrm{d}H$ is compactly supported. A **characteristic chord** of $\Sigma$ with endpoints in some specified Lagrangian submanifold $L$ of $X$ is a flow line of $\phi^H_t$ which starts and ends in $\Sigma \cap L$.

We are primarily interested in the case when the hypersurface $\Sigma$ satisfies the following condition:

**1.1** Definition. *A closed connected separating hypersurface $\Sigma$ is of* **virtual restricted contact type** *if there exists a primitive $\lambda$ of $\widetilde{\omega}$ such that:*

1. *For some (and hence any) Riemannian metric $g$ on $\Sigma$, there exists a constant $C = C(g) < \infty$ such that*
$$\sup_{x \in \mathsf{p}^{-1}(\Sigma)} |\lambda_x| \leq C,$$
*where $|\cdot|$ denotes the lift of $g$ to $\mathsf{p}^{-1}(\Sigma)$.*

2. *For some (and hence any) non-vanishing positively oriented section $\sigma$ of $\mathfrak{L}_\Sigma$, there exists a constant $\varepsilon = \varepsilon(\sigma) > 0$ such that*
$$\inf_{x \in \mathsf{p}^{-1}(\Sigma)} \lambda(\widetilde{\sigma}(x)) \geq \varepsilon,$$
*where $\widetilde{\sigma}$ denotes a lift of $\sigma$.*

If $\Sigma \subseteq X$ is of virtual restricted contact type then we denote by $\Omega^1_\Sigma(\widetilde{\omega}) \subseteq \Omega^1(\widetilde{X})$ the convex set of primitives $\lambda$ of $\widetilde{\omega}$ satisfying (1) and (2).

**1.2** Remark. *Note that asking $\Sigma$ to be of virtual restricted contact type includes as a special case the more well known condition when the symplectic form $\omega$ is exact, and the hypersurface $\Sigma$ is of* **restricted contact type** - *that is, there exists a primitive $\lambda$ of $\omega$ such that $\lambda|_\Sigma$ is a positive contact form on $\Sigma$.*

A **virtually restricted contact homotopy** of a pair $(H, \lambda)$, where $\Sigma := H^{-1}(0)$ is a hypersurface of virtual restricted contact type, with $H \in \mathscr{D}_0(\Sigma)$ and $\lambda \in \Omega^1_\Sigma(\widetilde{\omega})$, is a family $(H_s, \lambda_s)_{s \in (-\varepsilon, \varepsilon)}$ such that:

1. $(H_s)$ is a smooth family of uniformly compactly supported Hamiltonians such that $H_0 = H$, and such that $\Sigma_s := H_s^{-1}(0)$ is of virtual restricted contact type for each $s \in (-\varepsilon, \varepsilon)$, with $H_s \in \mathscr{D}_0(\Sigma_s)$;
2. $(\lambda_s)$ is a smooth family of 1-forms such that $\lambda_0 = \lambda$ and such that $\lambda_s \in \Omega^1_{\Sigma_s}(\widetilde{\omega})$ for each $s \in (-\varepsilon, \varepsilon)$, and such that the constants $\varepsilon(X_{H_s})$ from Condition (2) in Definition 1.1 may be taken to be independent of $s$.

**Assumptions on the Lagrangian:**

All Lagrangian submanifolds discussed in this paper are assumed to be connected, even if this is not explicitly stated. Suppose we are given a Lagrangian submanifold $L$ of $X$ that is $\pi_1$-injective. Since we assume that $\omega$ is symplectically aspherical and $c_1(TX) = 0$, the $\pi_1$-injectivity assumption implies that $\omega|_{\pi_2(X,L)} = c_1|_{\pi_2(X,L)} = 0$. Moreover it implies that the pullback $\mathsf{p}^{-1}(L) \subseteq \widetilde{X}$ can be written as a disjoint union of components each diffeomorphic to the universal cover $\widetilde{L}$ of $L$. In particular, each component of $\mathsf{p}^{-1}(L)$ is simply connected, and thus $\mathrm{H}^1(\mathsf{p}^{-1}(L); \mathbb{Z}) = 0$. Thus as $\omega|_L = 0$, if $\lambda \in \Omega^1(\widetilde{X})$ is a primitive of $\widetilde{\omega}$, we can find a smooth function $l : \mathsf{p}^{-1}(L) \to \mathbb{R}$ such that $\lambda|_{\mathsf{p}^{-1}(L)} = dl$.

We say that a $\pi_1$-injective Lagrangian $L$ is **virtually exact** if one can choose a primitive $\lambda$ of $\widetilde{\omega}$ and a function $l$ such that $\lambda|_{\mathsf{p}^{-1}(L)} = dl$, where $l \in C^\infty(\mathsf{p}^{-1}(L), \mathbb{R})$ is a **bounded** function. We denote by $\Omega^1_L(\widetilde{\omega}) \subseteq \Omega^1(\widetilde{X})$ the set of such primitives $\lambda$.

When talking about Lagrangian submanifolds $L$ of $X$, we shall always implicitly assume that $\star \in L$. Let $P(X, L)$ denote the set of smooth maps $x : [0,1] \to X$ with $x(0) \in L$ and $x(1) \in L$. Note that if $f : S^1 \to P(X, L)$ is a smooth loop then we may alternatively think of $f$ as a map $f : S^1 \times [0,1] \to X$ with $f(S^1 \times \{0,1\}) \subseteq L$.

Define

$$\Pi_L := \pi_1(X, \star)/\sim, \tag{1.1}$$

where for $\mathsf{a}, \mathsf{b} \in \pi_1(X, \star)$ we have $\mathsf{a} \sim \mathsf{b}$ if and only if there exists $\mathsf{c}_0, \mathsf{c}_1 \in \pi_1(L, \star)$ such that

$$\mathsf{a} = \iota_{L*}(\mathsf{c}_0) \cdot \mathsf{b} \cdot \iota_{L*}(\mathsf{c}_1)$$

(i.e. $\Pi_L = \pi_1(L) \backslash \pi_1(X)/\pi_1(L)$). It is not hard to see that $\Pi_L \cong \pi_0(P(X, L))$ (see for instance [37, Lemma 3.3.1]). Given $\alpha \in \Pi_L$, we denote by $P_\alpha(X, L)$ the connected component of $P(X, L)$ corresponding to $\alpha$, so that

$$P(X, L) = \bigoplus_{\alpha \in \Pi_L} P_\alpha(X, L).$$

Let us now fix for each $\alpha \in \Pi_L$ a smooth loop $x_\alpha : S^1 \to X$ with $x_\alpha(0) = \star$ such that $x_\alpha$ represents $\alpha$. It is convenient to choose these loops $x_\alpha$ so that $x_0(t) = \star$ for all $t$, and such that $x_\alpha(t) = x_{-\alpha}(1-t)$. Fix a point $\widetilde{\star} \in \widetilde{X}$ that projects onto $\star$, and for each $\alpha \in \Pi_L$ let $\widetilde{x}_\alpha : [0,1] \to \widetilde{X}$ denote the unique lift of $x_\alpha$ with $\widetilde{x}_\alpha(0) = \widetilde{\star}$. In particular, $\widetilde{x}_0(t) = \widetilde{\star}$ for all $t$.

Given $x \in P_\alpha(X, L)$, let us say that a map $\bar{x} : [0,1] \times [0,1] \to X$ is a **filling** of $x$ if $\bar{x}$ satisfies:

- $\bar{x}(0, t) = x(t)$,
- $\bar{x}(1, t) = x_\alpha(t)$, and
- $\bar{x}([0,1] \times \{0,1\}) \subseteq L$.

If $f : S^1 \to P(X, L)$ is a smooth loop then we may alternatively think of $f$ as a map $f : S^1 \times [0,1] \to X$ with $f(S^1 \times \{0,1\}) \subseteq L$. Let us consider the following condition:

**(A)** If $f : S^1 \to P_\alpha(X, L)$ is any smooth loop then $\int_{S^1\times[0,1]} f^*\omega = 0$.

Since $L$ is $\pi_1$-injective and $\omega|_{\pi_2(X)} = 0$, we have that $\omega|_{\pi_2(X,L)} = 0$ and thus Condition **A** is satisfied for the element $0 \in \Pi_L$. We denote by $\Pi_L^\omega \subseteq \Pi_L$ the set of classes $\alpha$ for which Condition **A** is satisfied.

When Condition **A** is satisfied we can define the **symplectic area functional** $\Omega : P_\alpha(X, L) \to \mathbb{R}$ by

$$\Omega(x) := \int_{[0,1]\times[0,1]} \bar{x}^*\omega, \tag{1.2}$$

where $\bar{x}$ is any filling of $x$.

The precise conditions under which we will define the Lagrangian Rabinowitz Floer homology is given by the following definition.

**1.3** Definition. *A triple $(\Sigma, L, \alpha)$ is called* **Rabinowitz admissible** *if:*

- *$\Sigma$ is a hypersurface of virtual restricted contact type,*
- *$L$ is a $\pi_1$-injective virtually exact Lagrangian submanifold and $\Omega^1_\Sigma(\widetilde{\omega}) \cap \Omega^1_L(\widetilde{\omega}) \neq \emptyset$,*
- *$\alpha \in \Pi_L^\omega$.*

*If $(\Sigma, L, \alpha)$ is a Rabinowitz admissible pair, $H \in \mathscr{D}_0(\Sigma)$ and $\lambda \in \Omega^1_\Sigma(\widetilde{\omega}) \cap \Omega^1_L(\widetilde{\omega})$, we say that a virtually restricted contact homotopy $(H_s, \lambda_s)_{s\in(-\varepsilon,\varepsilon)}$ of $(H, \lambda)$ is* **Rabinowitz admissible** *if it has the additional property that $\lambda_s \in \Omega^1_{\Sigma_s}(\widetilde{\omega}) \cap \Omega^1_L(\widetilde{\omega})$ for all $s \in (-\varepsilon, \varepsilon)$.*

**1.4** Remark. *If $\omega$ is exact and $\Sigma$ is of restricted contact type, then instead of assuming that $L$ is $\pi_1$-injective and virtually exact, and that $\Omega^1_\Sigma(\widetilde{\omega}) \cap \Omega^1_L(\widetilde{\omega}) \neq \emptyset$, we may alternatively assume that there exists a primitive $\lambda$ of $\omega$ such that $\lambda|_\Sigma$ is a positive contact form on $\Sigma$* **and** *$\lambda|_L = dl$ for some bounded function $l \in C^\infty(L, \mathbb{R})$. Morever in this setting Condition* **A** *is obviously always satisfied.*

**Leaf-wise intersection points:**

Recall that a subset $S$ is **displaceable** from a subset $S'$ of a symplectic manifold $X$ if there exists a compactly supported Hamiltonian diffeomorphism $\psi$ such that $\psi(S) \cap S' = \emptyset$. As we will see in Theorem A below, the Lagrangian Rabinowitz Floer homology detects whether $\Sigma$ is displaceable from $L$. A refinement of this question is to ask whether a given Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}_c(X, \omega)$ has a **Lagrangian leaf-wise intersection point**. Here is the definition.

**1.5** Definition. *Let $\psi \in \mathrm{Ham}_c(X, \omega)$ and fix any $H \in \mathscr{D}(\Sigma)$. We will say that a point $x \in \Sigma \cap L$ is a* **Lagrangian leaf-wise intersection point** *for $(\Sigma, L, \psi)$ if there exists $\tau \in \mathbb{R}$ such that*

$$\psi(\phi^H_\tau(x)) \in L.$$

*We denote by $\mathscr{LW}(\Sigma, L, \psi)$ the set of such points $x \in \Sigma \cap L$.*

Note that the set $\mathscr{LW}(\Sigma, L, \psi)$ is well defined , since if $H' \in \mathscr{D}(\Sigma)$ is another Hamiltonian then there exists a smooth function $f : \mathbb{R} \times \Sigma \to \mathbb{R}$ such that $\phi^{H'}_t(x) = \phi^H_{f(t,x)}(x)$ for all $(t, x) \in \mathbb{R} \times \Sigma$.

**1.6** Remark. *It is of interest to know whether $\tau$ is uniquely determined by $x$. This could fail if the leaf $\mathfrak{F}_\Sigma(x)$ is closed, or if $\#\psi(\mathfrak{F}_\Sigma(x)) \cap L > 1$. If $\dim X \geq 4$, then generically, neither of these things happen. More precisely, if $\Sigma$ is non-degenerate (as defined in Section 2.2) then for a generic choice of $\psi$, one has that if $x \in \mathscr{LW}(\Sigma, L, \psi)$ then $\mathfrak{F}_\Sigma(x)$ is not closed and $\#\psi(\mathfrak{F}_\Sigma(x)) \cap L = 1$. These statements are proved by arguing as in [9, Theorem 3.3] and [7, Lemma 8.2].*

Suppose now that $x$ is a Lagrangian leaf-wise intersection point for which the corresponding $\tau$ **is** uniquely determined by $x$. Choose any Hamiltonian $F \in C_c^\infty(S^1 \times X, \mathbb{R})$ that **generates** $\psi$ (so $\psi = \phi_1^F$), and consider the (not necessarily smooth) path $\zeta$ in $X$ which first flows from $x$ to $\phi_\tau^H(x)$ via $\phi_t^H$, and then flows from $\phi_\tau^H(x)$ to $\psi(\phi_\tau^H(x))$ via $\phi_t^F$. Although $\zeta$ depends on the choice of Hamiltonians $H$ and $F$, the class $\alpha \in \Pi_L$ does not. This is a standard argument, which uses the fact any 1-periodic compactly supported Hamiltonian function on $X$ has at least one contractible periodic orbit (i.e. the Arnold Conjecture holds on $(X, \omega)$). Details can be found in several places; see for instance [41, Proposition 3.1] or [32, Lemma 3.7] (the latter reference deals specifically with leaf-wise intersection points). In either case we say that the Lagrangian leaf-wise intersection point **belongs** to the class $\alpha \in \Pi_L$.

**Statement of Theorem A:**

We can now state the main theorem of Part I. The term "non-degenerate" that appears in Statement (4) will be explained in Section 2.2 below.

**Theorem A.** *Let $(X, \omega)$ denote a symplectically aspherical geometrically bounded connected symplectic manifold satisfying $c_1(TX) = 0$.*

*Let $(\Sigma, L, \alpha)$ denote a Rabinowitz admissible triple. Then:*

1. *The* **Lagrangian Rabinowitz Floer homology** $\mathrm{RFH}_*^\alpha(\Sigma, L, X)$ *is a well defined $\mathbb{Z}$-graded $\mathbb{Z}_2$-vector space which is invariant through Rabinowitz admissible virtually restricted contact homotopies.*
2. *If $\Sigma$ is transverse to $L$ and $\Sigma \cap L \neq \emptyset$ then if $\mathrm{RFH}_*^0(\Sigma, L, X) = 0$, there exists a characteristic chord of $\Sigma$ with endpoints in $L$ that represents $0 \in \Pi_L$.*
3. *If there exists $\psi \in \mathrm{Ham}_c(X, \omega)$ that has no Lagrangian leaf-wise intersection points belonging to $\alpha$ then $\mathrm{RFH}_*^\alpha(\Sigma, L, X) = 0$. In particular, if $\Sigma$ can be displaced from $L$ via a Hamiltonian diffeomorphism then $\mathrm{RFH}_*^\alpha(\Sigma, L, X) = 0$.*
4. *Suppose $\dim X \geq 4$. If $\mathrm{RFH}_*^\alpha(\Sigma, L, X)$ is infinite dimensional and the triple $(\Sigma, L, \alpha)$ is non-degenerate (which is satisfied generically), then a generic Hamiltonian diffeomorphism has infinitely many Lagrangian leaf-wise intersection points belonging to $\alpha$.*

**1.7** Remark. *This construction is also perfectly valid if instead of a single Lagrangian submanifold $L$, we work with a pair $L_0, L_1$ of virtually exact $\pi_1$-injective Lagrangian submanifolds which intersect each other transversely over $\Sigma$. In this case, we would need to make an additional assumption to ensure the relevant Maslov classes $\mu_{L_0,L_1}(\alpha)$ always vanish - which we need in order to be able to $\mathbb{Z}$-grade the Lagrangian Rabinowitz Floer homology. For instance, it would suffice to assume at least one of the subgroups $\iota_{L_i*}(\pi_1(L_i, \star))$ is a torsion subgroup of $\pi_1(X, \star)$ [37, Remark 3.3.2].*

**Twisted cotangent bundles:**

The second part of this paper computes the Lagrangian Rabinowitz Floer homology in a particular setting. Let $M$ denote a closed connected orientable $n$-dimensional manifold, where $n \geq 2$. Let $X := T^*M$ denote the cotangent bundle of $M$ and let $\pi : X \to M$ denote the footpoint map. Let $\mathsf{p} : \widetilde{M} \to M$ denote the universal cover of $M$, and $\widetilde{\pi} : \widetilde{X} \to \widetilde{M}$ its cotangent bundle.

We denote by $\lambda_0 \in \Omega^1(X)$ the **Liouville 1-form**, which in local coordinates $(q, p)$ on $X$ is written as $\lambda_0 = pdq$. The **canonical symplectic structure** on $X$ is the exact symplectic form $d\lambda_0$.

A **Dirac magnetic monopole** is simply a closed two-form $\sigma \in \Omega^2(M)$. This gives rise to a **twisted symplectic form** [14, 27] on the cotangent bundle

$$\omega_\sigma = d\lambda_0 + \pi^*\sigma.$$

We refer to $(X, \omega_\sigma)$ as a **twisted cotangent bundle**. In this paper $\sigma$ may or may not be exact, but we will always insist that $\sigma$ is **weakly exact**, that is, the lift $\widetilde{\sigma} := \mathsf{p}^*\sigma \in \Omega^2(\widetilde{M})$ is exact (this is equivalent to requiring that $\sigma|_{\pi_2(M)} = 0$). In this case we say that $(X, \omega_\sigma)$ is a **weakly exact twisted cotangent bundle**. In this case the symplectic manifold $(X, \omega_\sigma)$ fits into the framework of the previous chapter. Indeed, twisted cotangent bundles are always geometrically bounded and they always satisfy $c_1(TX) = 0$.

In fact, we will always make the additional assumption that $\widetilde{\sigma}$ admits a **bounded** primitive: there exists $\theta \in \Omega^1(\widetilde{M})$ such that $d\theta = \widetilde{\sigma}$ and such that

$$\sup_{q \in \widetilde{M}} |\theta_q| < \infty, \tag{1.3}$$

where the norm $|\cdot|$ is given by the lift of any Riemannian metric on $M$ to $\widetilde{M}$. In this case we say that $(X, \omega_\sigma)$ is a **hyperbolic twisted cotangent bundle**.

We are interested in regular energy levels of **Tonelli Hamiltonians** $H \in C^\infty(X, \mathbb{R})$. Here we recall that the classical Tonelli assumption means that $H$ is **fibrewise strictly convex** and **superlinear**. In other words, the second differential $\mathrm{d}^2(H|_{T_q^*M})$ of $H$ restricted to each tangent space $T_q^*M$ is positive definite, and

$$\lim_{|p| \to \infty} \frac{H(q,p)}{|p|} = \infty,$$

uniformly for $q \in M$. A closed connected oriented hypersurface $\Sigma \subseteq X$ is called a **Tonelli hypersurface** if there exists an autonomous Tonelli Hamiltonian $H \in C^\infty(X, \mathbb{R})$ for which 0 is a regular value of $H$ with $\Sigma = H^{-1}(0)$. One says that $H$ is a **defining Tonelli Hamiltonian** for $\Sigma$. A Tonelli hypersurface is called **Mañé supercritical** if there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ such that the **Mañé critical value** $c(H, \sigma)$ is negative (the Mañé critical value $c(H, \sigma)$ is defined in Section 3.1 below). Here we remark only that if $\widetilde{\sigma}$ admits no bounded primitives then $c(H, \sigma) = \infty$ for every Tonelli Hamiltonian $H$ - thus Mañé supercritical hypersurfaces can only exist when the symplectic manifold $(X, \omega_\sigma)$ is hyperbolic.

We shall see in Lemma 3.3 below that a Mañé supercritical hypersurface $\Sigma$ is always of virtual restricted contact type. More precisely, we can find a primitive $\theta$ of $\widetilde{\sigma}$ such that

$$\lambda_\theta := \widetilde{\lambda}_0 + \widetilde{\pi}^*\theta \tag{1.4}$$

satisfies the two conditions of Definition 1.1 (here $\widetilde{\lambda}_0$ denotes the Liouville 1-form on $\widetilde{X}$, and $\widetilde{\pi} : \widetilde{X} \to \widetilde{M}$ denotes the projection). We denote by $\Omega^1_\Sigma(\widetilde{\sigma})$ the set of such primitives $\theta$. Thus if $\theta \in \Omega^1_\Sigma(\widetilde{\sigma})$ then $\lambda_\theta \in \Omega^1_\Sigma(\widetilde{\omega}_\sigma)$.

Let us now discuss the Lagrangian submanifolds that we will work in. Suppose $S \subseteq M$ is a closed connected submanifold (we always assume our submanifolds are closed and connected, even if this is not explicitly stated). The **conormal bundle** $N^*S \subseteq X$ is defined to be the set

$$N^*S := \left\{ (q,p) \in X \,:\, q \in S,\, p|_{T_qS} = 0 \right\}.$$

This is a vector bundle over $S$ of rank the codimension of $S$. The Liouville form $\lambda_0$ vanishes on any conormal bundle; thus any conormal bundle is a Lagrangian submanifold of the symplectic manifold $(X, d\lambda_0)$. In fact, if $L \subseteq X$ is any Lagrangian submanifold that is a closed subset of $X$ and is such that $\lambda_0|_L = 0$ then $L = N^*S$ for some submanifold $S \subseteq M$ ([2, Proposition 2.1]).

In general though, $N^*S$ is **not** a Lagrangian submanifold of $(X, \omega_\sigma)$.

**1.8** Lemma. *Let $i : S \hookrightarrow M$ be a submanifold. Then $N^*S$ is a Lagrangian submanifold of $(X, \omega_\sigma)$ if and only if $\sigma|_S := i^*\sigma = 0$.*

When discussing cotangent bundles, it is more convenient to fix once and for all a point $\star \in M$ as a reference point, and then take $0_\star \in T^*_\star M$ to be our fixed reference point in $X$. When discussing submanifolds $S$ of $M$, we always implicitly assume that $\star \in S$ (note this implies $0_\star \in N^*S$). We also fix a point $\widetilde{\star} \in \widetilde{M}$ that projects onto $\star$. We denote by $P(M, S)$ the space of smooth paths $q : [0, 1] \to M$ with $q(0) \in S$ and $q(1) \in S$. We define $\Pi_S$ in exactly the same way as $\Pi_L$ was defined in (1.1), only with $M$ and $S$ replacing $X$ and $L$. Then $\Pi_S$ indexes the connected components of $P(M, S)$, and given $\alpha \in \Pi_S$ we let $P_\alpha(M, S)$ denote the corresponding connected component. If $x = (q, p) \in P(X, N^*S)$ then $q \in P(M, S)$, and under the obvious identification $\Pi_S \cong \Pi_{N^*S}$, if $x \in P_\alpha(X, N^*S)$ then $q \in P_\alpha(M, S)$. In particular, if we write our reference loops $x_\alpha$ as $(q_\alpha, p_\alpha)$, then $q_\alpha$ serves as a reference loop in $P_\alpha(M, S)$.

Next, observe that $N^*S$ is virtually exact if there exists a primitive $\theta$ of $\widetilde{\sigma}$ such that $\theta|_{\mathsf{p}^{-1}(S)} = ds$ for some **bounded** function $s \in C^\infty(\mathsf{p}^{-1}(S), \mathbb{R})$. We will also say $S$ is **virtually exact** if such primitives exist. We denote by $\Omega^1_S(\widetilde{\sigma})$ the set of such primitives $\theta$. Thus if $\theta \in \Omega^1_S(\widetilde{\sigma})$ then $\lambda_\theta$ (as defined in (1.4)) belongs to $\Omega^1_{N^*S}(\widetilde{\omega}_\sigma)$.

In general, if $\Sigma$ is a Mañé supercritical hypersurface and $S$ is a closed connected $\pi_1$-injective submanifold for which $\sigma|_S = 0$, there is no reason why one should have $\Omega^1_\Sigma(\widetilde{\sigma}) \cap \Omega^1_S(\widetilde{\sigma}) \neq \emptyset$ (see Example 21), and hence the Lagrangian Rabinowitz Floer homology may not be defined. We therefore introduce the stronger notion of a **Mañé supercritical pair**: a pair $(\Sigma, S)$ consisting of a Tonelli hypersurface and a closed connected $\pi_1$-injective submanifold satisfying $\sigma|_S = 0$ is called a **Mañé supercritical pair** if $\Omega^1_\Sigma(\widetilde{\sigma}) \cap \Omega^1_S(\widetilde{\sigma}) \neq \emptyset$. In Section 3.1 we will define another critical value $c(H, \sigma, S) \geq c(H, \sigma)$ which will have the property that if $\Sigma$ is a Tonelli hypersurface and $S$ is a closed connected $\pi_1$-injective submanifold satisfying $\sigma|_S = 0$ then $(\Sigma, S)$ is a Mañé supercritical pair if there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ satisfying $c(H, \sigma, S) < 0$. If $\pi_1(S)$ is finite then we have the equality $c(H, \sigma) = c(H, \sigma, S)$. Indeed, in this case the universal cover $\widetilde{S}$ of $S$ is compact, and hence **every** primitive $\theta$ of $\widetilde{\sigma}$ belongs to $\Omega^1_S(\widetilde{\sigma})$.

We now show that for a Mañé supercritical pair $(\Sigma, S)$, **every** class $\alpha \in \Pi_S$ belongs to $\Pi_S^{\omega_\sigma}$ (i.e. Condition **A** is satisfied for all $\alpha \in \Pi_S$).

**1.9** Lemma. *Suppose $(\Sigma, S)$ is a Mañé supercritical pair. Then for every path $f : S^1 \times [0, 1] \to X$ with $f(S^1 \times \{0, 1\}) \subseteq N^*S$, one has $\int_{S^1 \times [0,1]} f^*\omega_\sigma = 0$.*

**Proof.** The symplectic area functional $\Omega$ of $(X, \omega_\sigma)$ can be expressed as

$$\Omega = \Omega_0 + \pi^*\Omega_\sigma,$$

where

$$\Omega_0(x) := \int_0^1 x^*\lambda_0,$$

and $\Omega_\sigma$ is the **$\sigma$-area** defined by

$$\Omega_\sigma(q) := \int_{[0,1]\times[0,1]} \bar{q}^*\sigma,$$

where $\bar{q}$ is any filling of $q$ (i.e. any smooth map $\bar{q} : [0, 1] \times [0, 1] \to M$ with $\bar{q}(0, t) = q(t)$, $\bar{q}(1, t) = q_\alpha(t)$ and $\bar{q}([0, 1] \times \{0, 1\}) \subseteq S$).

It thus suffices to show that if $f : S^1 \times [0, 1] \to M$ satisfies $f(S^1 \times \{0, 1\}) \subseteq S$ then $\int_{S^1\times[0,1]} f^*\sigma = 0$. Fix a bounded primitive $\theta$ of $\widetilde{\sigma}$ with the property that $\theta|_{\mathsf{p}^{-1}(S)} = ds$ for some bounded function $s \in C^\infty(\mathsf{p}^{-1}(S), \mathbb{R})$. Consider $G := f_*(\pi_1(S^1 \times [0, 1])) \leq \pi_1(M)$. Then $G$ is **amenable**, since $\pi_1(S^1 \times [0, 1]) = \mathbb{Z}$, which is amenable. Then [36, Lemma 5.3] tells us that since $\|\theta\|_{L^\infty} < \infty$ and $\|s\|_{L^\infty} < \infty$, we can replace $\theta$ by a $G$-invariant primitive $\theta'$ of $\widetilde{\sigma}$,

and $s$ by a $G$-invariant function $s'$ satisfying $\theta'|_{\mathsf{p}^{-1}(S)} = ds'$. Thus $\theta'$ and $s'$ descend to define a primitive $\theta'' \in \Omega^1(S^1 \times [0,1])$ of $f^*\sigma$ and a function $s'' \in C^\infty(S^1 \times \{0,1\}, \mathbb{R})$ with the property that $\theta''|_{S^1\times\{0,1\}} = ds''$. Hence by Stokes' Theorem, $\int_{S^1\times[0,1]} f^*\sigma = 0$ as required. ■

In summary, we have the following theorem.

**1.10** THEOREM. *Let $(X, \omega_\sigma)$ denote a hyperbolic twisted cotangent bundle and let $S \subseteq M$ denote a closed connected $\pi_1$-injective submanifold such that $\sigma|_S = 0$. Let $\Sigma$ denote a Tonelli hypersurface in $X$, and assume that $(\Sigma, S)$ is a Mañé supercritical pair. Then the Lagrangian Rabinowitz Floer homology $\mathrm{RFH}_*^\alpha(\Sigma, N^*S, X)$ is well defined for all $\alpha \in \Pi_S$, and enjoys the properties stated in Theorem A.*

**1.11** REMARK. *If $\sigma$ is itself exact then the same result holds if $S$ is any closed connected submanifold (not necessarily $\pi_1$-injective) for which there exists a primitive $\theta$ of $\sigma$ such that $\lambda_0 + \pi^*\theta$ makes $\Sigma$ into a hypersurface of restricted contact type* **and** *such that $\theta|_S$ is exact. (see Remark 1.4).*

**Computing the Lagrangian Rabinowitz Floer homology:**

The aim of the second part of this paper is to actually compute the Lagrangian Rabinowitz Floer homology in the situation described by Theorem 1.10. This is done by extending to the Lagrangian setting the Abbondandolo-Schwarz [4] short exact sequence, which relates the Lagrangian Rabinowitz Floer chain complex to the **Morse complex** of an appropriate **free time action functional**. In our earlier paper [33] we extended the short exact sequence from [4] to the setting of twisted cotangent bundles, and the idea here is very similar. We state here only part of the main result (given as Theorem 3.15 below), as the full statement is rather convoluted.

**THEOREM B.** *Let $(X, \omega_\sigma)$ denote a hyperbolic twisted cotangent bundle and let $S \subseteq M$ denote a closed connected $\pi_1$-injective submanifold of dimension $0 \le d \le n$ such that $\sigma|_S = 0$. Let $\Sigma$ denote a Tonelli hypersurface in $X$, and assume that $(\Sigma, S)$ is a Mañé supercritical pair. Fix $0 \ne \alpha \in \Pi_S$. Then the Rabinowitz Floer homology $\mathrm{RFH}_*^\alpha(\Sigma, N^*S, X)$ satisfies*

$$\mathrm{RFH}_*^\alpha(\Sigma, N^*S, X) \cong \mathrm{H}_*(P_\alpha(M, S); \mathbb{Z}_2) \oplus \mathrm{H}^{-*+2d-n+1}(P_{-\alpha}(M, S); \mathbb{Z}_2). \tag{1.5}$$

*If $d < n/2$ then (1.5) continues to hold when $\alpha = 0$. If $d = n/2$ and $n \ge 4$ then (1.5) continues to hold when $\alpha = 0$ and $* \ne 0, 1$. In this case we can still compute the remaining two groups $\mathrm{RFH}_0^0(\Sigma, N^*S, X)$ and $\mathrm{RFH}_1^0(\Sigma, N^*S, X)$, but this is slightly harder to state concisely, and hence we defer the precise statement to Theorem 3.15 below. Finally if $d > n/2$ then (1.5) continues to hold when $\alpha = 0$ and $* \le -1$ or $* \ge 2d - n + 2$.*

**1.12** REMARK. *At present I am unable to complete the computation of the Lagrangian Rabinowitz Floer homology $\mathrm{RFH}_*^0(\Sigma, N^*S, X)$ when $d > n/2$ if $0 \le * \le 2d - n + 1$, or if $n = 2$ and $d = 1$ for $* = 0, 1$. I hope to rectify this gap in the future.*

*The statement (1.5) is certainly not always true for all $* \in \mathbb{Z}$. For instance, if $S = M$ so $N^*S = \mathfrak{o}M$, then if $\Sigma \cap \mathfrak{o}M = \emptyset$ we clearly have $\mathrm{RFH}_*^0(\Sigma, \mathfrak{o}M, X) = 0$ but $\mathrm{H}_*(P_0(M, M); \mathbb{Z}_2) \cong \mathrm{H}_*(M; \mathbb{Z}_2)$.*

By combining Theorem A and Theorem B we obtain various results on non-displaceablity and on the existence of Lagrangian leaf-wise intersections.

**COROLLARY C.** *Let $(X, \omega_\sigma)$ denote a hyperbolic twisted cotangent bundle and let $S \subseteq M$ denote a closed connected $\pi_1$-injective submanifold of dimension $0 \le d \le n$ such that $\sigma|_S = 0$. Let $\Sigma$ denote a Tonelli hypersurface in $X$, and assume that $(\Sigma, S)$ is a Mañé supercritical pair. Then:*

1. *$\Sigma$ cannot be displaced from $N^*S$ if any of the following hold:*

(a) $d < n/2$,

(b) $d = n/2$ and $n \geq 4$,

(c) $\#\Pi_S \geq 2$,

2. *For any $\psi \in \mathrm{Ham}_c(X, \omega_\sigma)$, one has $\mathscr{LW}(\Sigma, L, \psi) \neq \emptyset$ provided at least one of the conditions from (1) hold.*

3. *Moreover, if $\dim \mathrm{H}_*(P(M,S); \mathbb{Z}_2) = \infty$ and $\Sigma$ is non-degenerate, then for a generic $\psi \in \mathrm{Ham}_c(X, \omega_\sigma)$ there exist infinitely many Lagrangian leaf-wise intersection points for $\psi$ in $\Sigma$.*

**1.13** Remark. *We remind the reader again that if $\sigma$ is exact then both Theorem B and Corollary C continue to hold for any any closed connected submanifold $S$ (not necessarily $\pi_1$-injective) for which there exists a primitive $\theta$ of $\sigma$ such that $\lambda_0 + \pi^*\theta$ makes $\Sigma$ into a hypersurface of restricted contact type* **and** *such that $\theta|_S$ is exact (cf. Remark 1.11). With this in mind, I believe that Theorem B and Statements (2) and (3) of Corollary C in the Lagrangian case are new even in the case that $\sigma = 0$.*

A special case of the first statement of Corollary C is that it not possible to displace a Mañé supercritical hypersurface from a fibre $T_q^*M$ via a Hamiltonian diffeomorphism. However this doesn't tell us anything new, since the following stronger result is true for purely topological reasons.

**1.14** Lemma. *Suppose $\Sigma \subseteq X$ is a convex hypersurface (i.e. for all $q \in M$, $\Sigma \cap T_q^*M$ is a smooth closed hypersurface with positive definite 2nd fundamental form). Then $\Sigma$ cannot be displaced from any fibre $T_q^*M$.*

**Proof.** More generally, we prove that if $\psi : X \to X$ is any diffeomorphism isotopic to the identity and $\Sigma$ is any convex hypersurface then $\psi(\Sigma) \cap T_q^*M \neq \emptyset$ for every $q \in M$. Let $D(\Sigma)$ denote the compact domain in $X$ that $\Sigma$ bounds, and for each $q \in M$, let $e(q)$ denote the barycentre of the convex set $D(\Sigma) \cap T_q^*M$. This defines a smooth section $e : M \to X$. Let $\psi_t : X \to X$ denote an isotopy from $\psi_0 = \mathbb{1}$ to $\psi_1 = \psi$, and let $f_t : M \to M$ denote the composite map $f_t := \pi \circ \psi_t \circ e$. Then $f_0 = \mathbb{1}$; in particular $f_0$ has degree 1. If there exists $q \in M$ such that $\psi(\Sigma) \cap T_q^*M = \emptyset$ then necessarily $\psi(D(\Sigma)) \cap T_q^*M = \emptyset$, and hence $f_1$ cannot be surjective. This contradicts the fact that $f_1$ should also have degree 1. ■

I do not believe the corresponding result about the existence of leaf-wise intersections can be proved topologically however. Here is another a simple example where the second statement of Corollary C may be applied.

**1.15** Example. *Take $M = S^2$ and $S$ the equatorial $S^1$. Take $\sigma = 0$ and $\Sigma = S_g^*S^2$, the unit cotangent bundle for some metric $g$ on $S^2$. This fits into the framework where Theorem B applies (see Remark 1.13). Note that we cannot compute $\mathrm{RFH}_*(S_gS^2, N^*S^1, T^*S^2)$ for $* = 0, 1$. However, if we fix a point $q \in S^1$ then it is easily checked that the inclusion $\{q\} \hookrightarrow S^1$ induces an injection $\mathrm{H}_*(P(S^2, q); \mathbb{Z}_2) \to \mathrm{H}_*(P(S^2, S^1); \mathbb{Z}_2)$. Since $\dim \mathrm{H}_*(P(S^2, q); \mathbb{Z}_2) = \infty$, we deduce that if $g$ is bumpy[1], then a generic Hamiltonian diffeomorphism $\psi \in \mathrm{Ham}_c(T^*S^2, d\lambda_0)$ has infinitely many Lagrangian leaf-wise intersection points.*

**1.16** Remark. *In fact, work of Kang [30] shows that*

$$\mathrm{H}_m(P(S^2, S^1); \mathbb{Z}_2) \cong \bigoplus_{i+j+k=m} \mathrm{H}_i(P(S^2, q); \mathbb{Z}_2) \otimes \mathrm{H}_j(S^1; \mathbb{Z}_2) \otimes \mathrm{H}_k(S^1; \mathbb{Z}_2).$$

*Moreover in [30] Kang uses this computation together with Example 1.15 to obtain the existence of a certain periodic orbit in the* **regularized planar circular 3-body problem**.

[1] This just means that $S_g^*S^2$ is non-degenerate.

**1.17** Remark. *There is an exciting work in progress by C. Bounya [16] on Lagrangian Rabinowitz Floer homology. He independently proves a version of Theorem A, and then discusses the relationship between Lagrangian Rabinowitz Floer homology and wrapped Floer homology, in a similar vein to Cieliebak, Frauenfelder and Oancea's result [19] which relates Rabinowitz Floer homology with symplectic homology. His results will include (amongst other things) the computation from Theorem B, at least in the case of standard cotangent bundles.*

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# 2 Lagrangian Rabinowitz Floer homology

**Notation and sign conventions:**

- We denote by $\overline{\mathbb{R}}$ the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, and we write $\mathbb{R}^+ := (0, \infty)$ and $\mathbb{R}^- := (-\infty, 0)$.
- We refer to both points in $X$ and paths on $X$ by the letter $x$. It should hopefully be clear from the context whether $x$ refers to a point in $X$ or a path in $X$. In addition we use the convention that if $x \in X$ is a point, $\underline{x} : [0, 1] \to X$ denotes the constant path $\underline{x}(t) = x$.
- A lot of the time we will work with the product space $P(X, L) \times \mathbb{R}$. We will often use the single letter $\zeta$ to refer to a pair $(x, \tau) \in P(X, L \times \mathbb{R}$. By an abuse of notation, if $(x, \tau) \in P(X, L) \times \mathbb{R}$ with $\tau \neq 0$ we shall also refer to by $\zeta$ the path $\zeta : [0, \tau] \to X$ defined by $\zeta(t) := x(t/\tau)$. This identification obviously breaks down when $\tau = 0$. However it will turn out that we will only ever be interested in points of the form $(x, 0)$ when $x = \underline{x}$ is constant. In this case we can still identify the pair $(\underline{x}, 0)$ with the constant path $\zeta : \mathbb{R} \to X$ defined by $\zeta(t) = x$. In general it should hopefully always be clear as to whether $\zeta$ refers to the pair $(x, \tau)$ or to the path $x(t/\tau)$ [resp. the constant path $\zeta : t \mapsto x$ when $(x, \tau) = (\underline{x}, 0)$].
- We use the sign convention that an almost complex structure $J$ on a symplectic manifold $(X, \omega)$ is **$\omega$-compatible** if $g_J = \langle \cdot, \cdot \rangle_J := \omega(J\cdot, \cdot)$ is a Riemannian metric on $X$. We denote by $\mathscr{J}(X, \omega)$ the set of all $\omega$-compatible almost complex structures on $X$.
- Given a family $\mathbf{J} = (J_t)_{t\in[0,1]} \subseteq \mathscr{J}(X, \omega)$, and $x \in P(X, L)$ we use the special notation $\langle\langle \cdot, \cdot \rangle\rangle_{\mathbf{J}}$ to denote the inner product on $C^\infty(x^*TX) \times \mathbb{R}$ defined by

$$\langle\langle (\xi, h), (\xi', h') \rangle\rangle_{\mathbf{J}} := \int_0^1 \langle \xi(t), \xi'(t) \rangle_{J_t} \, dt + hh'.$$

- The **symplectic gradient** $X_H \in \mathrm{Vect}(X)$ of a Hamiltonian $H : X \to \mathbb{R}$ is defined by $i_{X_H}\omega = -\mathrm{d}H$.

## 2.1 The Rabinowitz action functional

Throughout Section 2 we will work in the general setting described at the beginning of the Introduction. We shall return to twisted cotangent bundles in Section 3. Thus assume throughout this section that $(X, \omega)$ is a symplectic manifold satisfying the assumptions outlined on page 1. We now define the **Rabinowitz action functional**, introduced originally by Rabinowitz in [38]. Note at this point we do **not** assume the Hamiltonian $H$ in Definition 2.1 below lies in $\mathscr{D}_0(\Sigma)$ for some hypersurface $\Sigma$ of virtual restricted contact type (although this will be the case shortly). Throughout this section we fix an element $\alpha \in \Pi_L^\omega$.

**2.1** Definition. *Let $H \in C^\infty(X, \mathbb{R})$, and assume that $0$ is a regular value of $H$. The* **Rabinowitz action functional** $\mathscr{A}_H : P_\alpha(X, L) \times \mathbb{R} \to \mathbb{R}$ *is defined by*

$$\mathscr{A}_H(x, \tau) = \Omega(x) - \tau \int_0^1 H(x)dt,$$

*where $\Omega$ is the symplectic area functional from* (1.2).

An easy computation shows that given a tangent vector $(\xi, h) \in C^\infty(x^*TX) \times \mathbb{R}$, we have

$$\begin{aligned} \mathrm{d}\mathscr{A}_H(x,\tau)(\xi,h) &= \int_0^1 \omega(\dot{x}, \xi) - \tau \mathrm{d}H(x)(\xi)dt - h\int_0^1 H(x)dt \\ &= \int_0^1 \omega(\dot{x} - \tau X_H(x), \xi)dt - h \int_0^1 H(x)dt, \end{aligned}$$

and thus the critical points of $\mathscr{A}_H$ are pairs $(x, \tau)$ such that

$$\dot{x} = \tau X_H(x(t)) \quad \text{for all } t \in [0,1],$$

$$\int_0^1 H(x)dt = 0.$$

Since $H$ is invariant under its Hamiltonian flow, the second equation implies

$$H(x(t)) = 0 \quad \text{for all } t \in [0,1],$$

and so

$$x([0,1]) \subseteq H^{-1}(0).$$

Thus if we denote by $\mathrm{Crit}(\mathscr{A}_H)$ the set of critical points of $\mathscr{A}_H$ then $(x,\tau) \in P_\alpha(X, L) \times \mathbb{R}$ belongs to $\mathrm{Crit}(\mathscr{A}_H)$ if and only if

$$\dot{x} = \tau X_H(x), \quad x([0,1]) \subseteq H^{-1}(0).$$

Note that if $(x, \tau) \in \mathrm{Crit}(\mathscr{A}_H)$ then

$$\mathscr{A}_H(x,\tau) = \Omega(x). \tag{2.1}$$

Given $-\infty < a < b < \infty$, denote by

$$\mathrm{Crit}(\mathscr{A}_H)_a^b := \{(x,\tau) \in \mathrm{Crit}(\mathscr{A}_H) \, : \, a \le \mathscr{A}_H(x,\tau) \le b\}.$$

We always implicitly assume when referring to action windows that the endpoints $a$ and $b$ are not critical values of $\mathscr{A}_H$. When it is necessary to specify which connected component of $P(X, L) \times \mathbb{R}$ we are working on, we write $\mathrm{Crit}^\alpha(\mathscr{A}_H)$ for the set of critical points of $\mathscr{A}_H$ in $P_\alpha(X, L) \times \mathbb{R}$.

It will be useful sometimes to consider for fixed $\tau \in \mathbb{R}$ the **fixed period action functional** $\mathscr{A}_H^\tau : P_\alpha(X, L) \to \mathbb{R}$ defined by

$$\mathscr{A}_H^\tau(x) := \mathscr{A}_H(x, \tau). \tag{2.2}$$

Note that

$$\mathrm{d}\mathscr{A}_H^\tau(x)(\xi) = \mathrm{d}\mathscr{A}_H(x,\tau)(\xi, 0),$$

and hence if $(x,\tau) \in \mathrm{Crit}(\mathscr{A}_H)$ then $x \in \mathrm{Crit}(\mathscr{A}_H^\tau)$.

Suppose $\mathbf{J} = (J_t)_{t \in [0,1]} \subseteq \mathscr{J}(X, \omega)$. We let $\nabla_{\mathbf{J}} \mathscr{A}_H$ denote the gradient of $\mathscr{A}_H$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\mathbf{J}}$, so that

$$\nabla_{\mathbf{J}} \mathscr{A}_H(x,\tau) = \left( \begin{array}{c} J_t(x)(\dot{x} - \tau X_H(x)) \\ -\int_0^1 H(x)dt \end{array} \right).$$

Given $(x,\tau) \in \mathrm{Crit}(\mathscr{A}_H)$ let us denote by

$$\nabla^2_{\mathbf{J}} \mathscr{A}_H(x,\tau) : W^{1,r}(x^*TX) \oplus \mathbb{R} \to L^r(x^*TX) \oplus \mathbb{R}$$

(for some fixed $r \geq 2$) the operator obtained by linearizing $\nabla_{\mathbf{J}}\mathscr{A}_H(x,\tau)$, that is,

$$\nabla^2_{\mathbf{J}}\mathscr{A}_H(x,\tau)(\xi,h) := \frac{\partial}{\partial s}\Big|_{s=0}\nabla_{\mathbf{J}}\mathscr{A}_H(x_s,\tau_s),$$

where $(x_s,\tau_s)_{s\in(-\varepsilon,\varepsilon)} \subseteq P_\alpha(X,L)\times\mathbb{R}$ satisfies

$$\frac{\partial}{\partial s}\Big|_{s=0}(x_s,\tau_s) = (\xi,h).$$

One computes that:

$$\nabla^2_{\mathbf{J}}\mathscr{A}_H(x,\tau)\begin{pmatrix}\xi\\ h\end{pmatrix} := \begin{pmatrix} J_t(x)\nabla_t\xi + (\nabla_\xi J_t)\dot{x} - \tau\nabla_\xi\nabla_{J_t}H - h\nabla_{J_t}H \\ -\int_0^1 \mathrm{d}H(\xi)dt\end{pmatrix}, \tag{2.3}$$

where $\nabla_{J_t}H$ denotes the gradient of $H$ with respect to the metric $g_{J_t}$.

We also denote by $\nabla^2_{\mathbf{J}}\mathscr{A}^\tau_H(x)$ the Hessian of the fixed period action functional $\mathscr{A}^\tau_H$ at a critical point $x$ of $\mathscr{A}^\tau_H$:

$$\nabla^2_{\mathbf{J}}\mathscr{A}^\tau_H(x)(\xi) := J_t(x)\nabla_t\xi + (\nabla_\xi J_t)\dot{x} - \tau\nabla_\xi\nabla_{J_t}H.$$

Thus if $(x,\tau)\in\mathrm{Crit}(\mathscr{A}_H)$ we have

$$\nabla^2_{\mathbf{J}}\mathscr{A}_H(x,\tau)\begin{pmatrix}\xi\\ h\end{pmatrix} = \begin{pmatrix}\nabla^2_{\mathbf{J}}\mathscr{A}^\tau_H(x) & -\nabla_{J_t}H(x)\\ -\nabla_{J_t}H(x)^* & 0\end{pmatrix}\begin{pmatrix}\xi\\ h\end{pmatrix}.$$

**2.2** Definition. *A* **gradient flow line** *of $(H,\mathbf{J})$ is a smooth map $u=(x,\tau):\mathbb{R}\to P_\alpha(X,L)\times\mathbb{R}$ such that*

$$\partial_s u + \nabla_{\mathbf{J}}\mathscr{A}_H(u(s)) = 0.$$

*In components this reads:*

$$\partial_s x + J_t(x)(\partial_t x - \tau X_H(x)) = 0;$$

$$\partial_s\tau - \int_0^1 H(x)dt = 0.$$

*Given $-\infty < a < b < \infty$, denote by $\mathscr{M}^\alpha(H,\mathbf{J})^b_a$ the set of gradient flow lines $u$ of $(H,\mathbf{J})$ that satisfy $a \leq \mathscr{A}_H(u(s)) \leq b$ for all $s\in\mathbb{R}$.*

It will often be useful to let both $H$ and $\mathbf{J}$ depend on a parameter $s\in\mathbb{R}$. Suppose $(H_s)_{s\in\mathbb{R}} \subseteq C^\infty(X,\mathbb{R})$ is a smooth family of Hamiltonians, which is **asymptotically constant** in the sense that there exist $H_\pm \in C^\infty(X,\mathbb{R})$ such that $H_s = H_-$ for $s \ll 0$ and $H_s = H_+$ for $s \gg 0$. Assume that 0 is a regular value of both $H_-$ and $H_+$. Similarly, suppose we are given a family $(\mathbf{J}_s = (J_{s,t}))_{s\in\mathbb{R}} \subseteq \mathscr{J}(X,\omega)$ of almost complex structures which is also asymptotically constant - that is, there exist families $\mathbf{J}_\pm = (J_{\pm,t}) \subseteq \mathscr{J}(X,\omega)$ of almost complex structures $\mathbf{J}_s = \mathbf{J}_-$ for $s \ll 0$ and $\mathbf{J}_s = \mathbf{J}_+$ for $s \gg 0$. It then makes sense to study the **$s$-dependent equation**

$$\partial_s u + \nabla_{\mathbf{J}_s}\mathscr{A}_{H_s}(u(s)) = 0,$$

and given $-\infty < a < b < \infty$, we denote by $\mathscr{M}^\alpha(H_s,\mathbf{J}_s)^b_a$ the set of smooth maps $u=(x,\tau)$ that satisfy this equation together with the asymptotic conditions

$$\lim_{s\to-\infty}\mathscr{A}_{H_s}(u(s)) \leq b, \quad \lim_{s\to\infty}\mathscr{A}_{H_s}(u(s)) \geq a.$$

Note that if $(H_s,\mathbf{J}_s) = (H,\mathbf{J})$ does not depend on $s$ then $\mathscr{M}^\alpha(H_s,\mathbf{J}_s)^b_a = \mathscr{M}^\alpha(H,\mathbf{J})^b_a$.

Given a gradient flow line $u$, we denote by

$$D_u : W^{1,r}(x^*TX)\oplus W^{1,r}(\mathbb{R},\mathbb{R}) \to L^r(x^*TX)\oplus L^r(\mathbb{R},\mathbb{R})$$

the vertical derivative of the gradient flow equation at a solution $u$, given by

$$D_u\begin{pmatrix}\xi\\ h\end{pmatrix} := \begin{pmatrix}\nabla_s\xi + J_t(x)\nabla_t\xi + (\nabla_\xi J_t)\partial_t x - \tau\nabla_\xi\nabla_{J_t}H - h\nabla_{J_t}H\\ \partial_s h - \int_0^1 \mathrm{d}H(x)(\xi)dt\end{pmatrix}. \tag{2.4}$$

Note that in the special case where $u(s) = (x,\tau)\in\mathrm{Crit}(\mathscr{A}_H)$ we have $D_u = \nabla^2_{\mathbf{J}}\mathscr{A}_H(x,\tau)$.

## 2.2 Non-degeneracy

We next define what it means for a critical point of $\mathscr{A}_H$ to be non-degenerate. Denote by $\mathscr{P}_\alpha(X,L)$ the Sobolev completion of $P_\alpha(X,L)$ with respect to the Sobolev $W^{1,2}$-norm. Let $\mathscr{W}$ denote the Hilbert manifold $\mathscr{P}_\alpha(X,L)\times\mathbb{R}$, and let $\mathscr{E}\to\mathscr{W}$ denote the Hilbert bundle whose fibre over $(x,\tau)\in\mathscr{W}$ is given by

$$\mathscr{E}_{(x,\tau)} := L^2(x^*TX)\times\mathbb{R}.$$

We define a section

$$s = s(H,\mathbf{J}) : \mathscr{W}\to\mathscr{E}$$

by

$$s(x,\tau) := \nabla_{\mathbf{J}}\mathscr{A}_H(x,\tau).$$

We denote by

$$Ds(\zeta) : T_\zeta\mathscr{W}\to\mathscr{E}_\zeta$$

the vertical derivative of $s$ at a zero $\zeta=(x,\tau)\in s^{-1}(0)$.

Fix a critical point $\zeta=(x,\tau)\in\mathrm{Crit}(\mathscr{A}_H)$ with $\tau\neq 0$. We say that $\zeta$ is **non-degenerate** if $Ds(\zeta)$ is surjective. We say that the critical points $\zeta\in(\underline{H^{-1}(0)\cap L},0)\subseteq\mathrm{Crit}(\mathscr{A}_H)$ are **non-degenerate** if 0 is a regular value of $H$, and $H^{-1}(0)$ is transverse to $L$.

Let us denote by $C^\infty_{\mathrm{reg},L,\alpha}(X,\mathbb{R})$ the set of smooth functions $H:X\to\mathbb{R}$ for which every critical point $(x,\tau)$ of $\mathscr{A}_H : P_\alpha(X,L)\times\mathbb{R}\to\mathbb{R}$ is non-degenerate.

It is well known that if every critical point in $\mathrm{Crit}(\mathscr{A}_H)_a^b$ is non-degenerate then for any choice of $\mathbf{J}=(J_t)\subseteq\mathscr{J}(X,\omega)$, every element $u\in\mathscr{M}^\alpha(H,\mathbf{J})_a^b$ is asymptotically convergent at each end to elements of $\mathrm{Crit}(\mathscr{A}_H)_a^b$. That is, the limits

$$\lim_{s\to\pm\infty} u(s,t) = (x_\pm(t),\tau_\pm),\qquad \lim_{s\to\infty}\partial_t u(s,t)=0,$$

exist, and the convergence is uniform in $t$, and the limits $(x_\pm,\tau_\pm)$ belong to $\mathrm{Crit}(\mathscr{A}_H)_a^b$ (see for instance [40]). Thus a gradient flow line $u$ defines a map $\overline{\mathbb{R}}\times[0,1]\to X\times\mathbb{R}$, which we will continue to denote by $u$.

Moreover, if $E_{\mathbf{J}}(u)$ denotes the **energy** of a gradient flow line:

$$E_{\mathbf{J}}(u) := \int_{-\infty}^{\infty}\|\partial_s u(s)\|^2_{\mathbf{J}}\,ds,$$

then if $u\in\mathscr{M}^\alpha(H,\mathbf{J})_a^b$ is asymptotically convergent to $\zeta_\pm\in\mathrm{Crit}(\mathscr{A}_H)_a^b$ then

$$E_{\mathbf{J}}(u) = \mathscr{A}_H(\zeta_-)-\mathscr{A}_H(\zeta_+),$$

and hence $E_{\mathbf{J}}(u)\in(0,b-a)$.

Let us now fix a hypersurface $\Sigma\subseteq X$ that is of virtual restricted contact type and which is transverse to $L$. To avoid trivialities[2] let us assume $\Sigma\cap L\neq\emptyset$.

Fix $\alpha\in\Pi_L$. Denote by $\mathscr{C}_\alpha(\Sigma,L)$ the set of characteristic chords of $\Sigma$ with endpoints in $L$ belonging to $\alpha$. If $H\in\mathscr{D}(\Sigma)$ then we can parametrize the elements of $\mathscr{C}_\alpha(\Sigma,L)$ as smooth paths $\zeta:[0,\tau]\to\Sigma$ satisfying $\dot\zeta = X_H(\zeta)$.

We will now associate to each $\zeta\in\mathscr{C}_\alpha(\Sigma,L)$ its **nullity**

$$n(\zeta) := \dim\,\mathrm{d}\phi^H_\tau(\zeta(0))(T_{\zeta(0)}L)\cap T_{\zeta(\tau)}L).$$

We say $\zeta\in\mathscr{C}_\alpha(\Sigma,L)$ is **non-degenerate** if $n(\zeta)=0$, and we say that the triple $(\Sigma,L,\alpha)$ is **non-degenerate** if every element $\zeta\in\mathscr{C}_\alpha(\Sigma,L)$ is non-degenerate.

[2]It is clear that the construction of Lagrangian Rabinowitz Floer homology still works when $\Sigma\cap L=\emptyset$ (it is trivially zero).

**2.3** Definition. *Fix $H \in \mathscr{D}(\Sigma)$. Given a characteristic chord $\zeta \in \mathscr{C}_\alpha(\Sigma, L)$ of length $\tau > 0$, we say that $\zeta$ admits a* **chord box** *if there exists a smooth (in s) family $(\zeta_s)_{s\in(-\varepsilon,\varepsilon)}$ of chords of $X_H$ with endpoints in $L$ such that*

$$\zeta_s : [0, \tau(s)] \to X,$$
$$H(\zeta_s(t)) = e(s),$$

*where $e : (-\varepsilon, \varepsilon) \to \mathbb{R}$ satisfies $e(0) = 0$, with $\zeta_0 = \zeta$. We say that the chord box is* **non-degenerate** *if $\tau'(0) \neq 0$ and $e'(0) \neq 0$. In this case we define the* **correction term** *associated to $\zeta$*

$$\chi(\zeta) := \operatorname{sign}\left(-\frac{e'(0)}{\tau'(0)}\right).$$

**2.4** Remark. *A priori, it would appear that the correction term $\chi(\zeta)$ could depend on the choice of chord box $(\zeta_s)$. In fact, this is not the case, as is proved in [34, Lemma 5.12].*

If $\zeta$ is a non-degenerate chord then $\zeta$ admits a non-degenerate chord box $(\zeta_s)$. See [8, Proposition B.1] for a proof of the existence of $(\zeta_s)$, and the fact that $\tau'(0) \neq 0$. To see that one also has $e'(0) \neq 0$, we argue as follows. Let $x \in P_\alpha(X, L)$ be defined by $x(t) = \zeta(\tau t)$. Let $\xi \in C^\infty(x^*TX)$ be defined by $\xi(t) := \frac{\partial}{\partial s}\Big|_{s=0} \zeta_s(t\tau(s))$. Then a direct computation shows that

$$\nabla^2_{\mathbf{J}} \mathscr{A}_H(x, \tau)(\xi, \tau'(0)) = (0, e'(0)),$$

where $\nabla^2_{\mathbf{J}} \mathscr{A}_H(x, \tau)$ was defined in (2.3). Since $\zeta$ is non-degenerate, $\nabla^2_{\mathbf{J}} \mathscr{A}_H(x, \tau)$ is injective (see the beginning of the proof of Lemma 2.5 below), and hence we must have $e'(0) \neq 0$.

The following lemma explains the relation between the various notions of non-degeneracy we have introduced.

**2.5** Lemma. **(Non-degeneracy)**

1. *Suppose $\Sigma$ is regular hypersurface which is transverse to $L$. Fix $H \in \mathscr{D}(\Sigma)$. Then $(x, \tau) \in \mathrm{Crit}^\alpha(\mathscr{A}_H)$ with $\tau \neq 0$ is non-degenerate as a critical point of $\mathscr{A}_H$ if and only if the corresponding element $\zeta(t) = x(t/\tau)$ is a non-degenerate element of $\mathscr{C}_\alpha(\Sigma, L)$. Thus if $(\Sigma, L, \alpha)$ is a non-degenerate triple then $\mathscr{D}_0(\Sigma) \subseteq C^\infty_{\mathrm{reg},L,\alpha}(X, \mathbb{R})$.*
2. *If $\alpha \neq 0$ and $H \in C^\infty_{\mathrm{reg},L,\alpha}(X, \mathbb{R})$ then $\mathscr{A}_H : P_\alpha(X, L) \times \mathbb{R} \to \mathbb{R}$ is a Morse-Bott functional and $\mathrm{Crit}^\alpha(\mathscr{A}_H)$ consists of an isolated collection of points. If $\alpha = 0$ and $H \in C^\infty_{\mathrm{reg},L,0}(X, \mathbb{R})$ then $\mathscr{A}_H : P_0(X, L) \times \mathbb{R} \to \mathbb{R}$ is a Morse-Bott functional and $\mathrm{Crit}^0(\mathscr{A}_H)$ consists of an isolated collection of points and a copy of $\Sigma \cap L$.*
3. *For all $\alpha \in \Pi_L$ the sets $C^\infty_{\mathrm{reg},L,\alpha}(X, \mathbb{R})$ are generic subsets of $C^\infty(X, \mathbb{R})$.*

**Proof.** We will only sketch the ideas behind the proof. Statement (1) follows directly from the definitions, and Statement (3) is by now a very standard argument. To prove (2) first note that as $\nabla^2_{\mathbf{J}} \mathscr{A}_H(x, \tau)$ is self-adjoint, it is Fredholm of index 0 and hence it is injective if and only if it is surjective. The fact that elements $(x, \tau) \in \mathrm{Crit}(\mathscr{A}_H)$ with $\tau \neq 0$ are isolated follows easily from the injectivity of $\nabla^2_{\mathbf{J}} \mathscr{A}_H(x, \tau)$. It remains to check that the set $(\underline{\Sigma \cap L}, 0)$ is a Morse-Bott component of $\mathrm{Crit}(\mathscr{A}_H)$. Since $\Sigma$ is transverse to $L$ by assumption, it suffices to show that for all $x \in \Sigma \cap L$ we have

$$\ker \nabla^2_{\mathbf{J}} \mathscr{A}_H(\underline{x}, 0) = T_x(\Sigma \cap L) \times \{0\} \subseteq T_{(\underline{x},0)}(P_0(X, L) \times \mathbb{R}). \tag{2.5}$$

If $x \in \Sigma \cap L$, an element $(\xi, h) \in T_{(\underline{x},0)}(P_0(X, L) \times \mathbb{R})$ is in the kernel of $\nabla^2_{\mathbf{J}} \mathscr{A}_H(\underline{x}, 0)$ if and only if $(\xi, h)$ solves the equations:

$$\dot{\xi} = hX_H(x),$$
$$\int_0^1 \mathrm{d}H(x)(\xi)dt = 0.$$

Integrating the first equation, we discover that $\xi(t) = \xi(0) + hX_H(x)$. Since $\xi(0) \in T_xL$ and $X_H(x) \notin T_xL$ as $\Sigma$ is transverse to $L$ we must have $h = 0$. Thus $\xi(t) = \xi(0)$ is constant. The second equation tells us that $\mathrm{d}H(x)(\xi(0)) = 0$, and hence $\xi(0) \in T_x\Sigma$. The proof is complete. ■

## 2.3 Compactness

Recall that an $\omega$-compatible almost complex structure $J$ is **geometrically bounded** if the corresponding Riemannian metric $\omega(J\cdot,\cdot)$ is complete, has bounded sectional curvature and has injectivity radius bounded away from zero. By our initial assumption on $X$ such almost complex structures exist; let us fix once and for all such an almost complex structure $J_{\mathrm{gb}}$. We denote by $\mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}}) \subseteq \mathscr{J}(X,\omega)$ the set of almost complex structures $J \in \mathscr{J}(X,\omega)$ for which there exists a compact set $K \subseteq X$ (depending on $J$) such that $J = J_{\mathrm{gb}}$ on $X \backslash K$. Since in general it is unknown whether the set of all geometrically bounded almost complex structures is connected, it is possible that everything we do will depend on our initial choice of geometrically bounded almost complex structure $J_{\mathrm{gb}}$. The general consensus however seems to be that this is unlikely. Regardless, we will ignore this subtlety throughout.

The following two compactness results are key to everything that follows. The first result is for gradient flow lines of a pair $(H,\mathbf{J})$; the second result is for $s$-dependent trajectories. These results were originally proved in the periodic case for hypersurfaces of restricted contact type in [18]. A full proof in this setting can be found in [34]. We remark that it is these results where the hypothesis that $\Sigma$ is of virtual restricted contact type is used, and where we use the fact that $\Omega^1_\Sigma(\widetilde{\omega}) \cap \Omega^1_L(\widetilde{\omega}) \neq \emptyset$ holds.

**2.6** Theorem. *Assume $(\Sigma,L,\alpha)$ is a non-degenerate Rabinowitz admissible triple. Let $H \in \mathscr{D}_0(\Sigma)$ and $\mathbf{J} = (J_t) \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$. Fix $-\infty < a < b < \infty$ and suppose $(u_m = (x_m,\tau_m))_{m\in\mathbb{N}} \subseteq \mathscr{M}^\alpha(H,\mathbf{J})_a^b$. Then for any sequence $(s_m) \subseteq \mathbb{R}$, the reparametrized sequence $u_m(\cdot + s_m)$ has a subsequence which converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times[0,1],X) \times C^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$.*

**2.7** Theorem. *Assume $(\Sigma_\pm,L,\alpha)$ are both non-degenerate Rabinowitz admissible triples. Fix Hamiltonians $H_\pm \in \mathscr{D}_0(\Sigma_\pm)$ and suppose that there exists a Rabinowitz admissible virtually contact homotopy $(H_s,\lambda_s)_{s\in\mathbb{R}}$ such that $(H_s,\lambda_s) = (H_-,\lambda_-)$ for $s \ll 0$ and $(H_s,\lambda_s) = (H_+,\lambda_+)$ for $s \gg 0$, and such that $H_s$ has compact support uniformly in $s$. Fix $\mathbf{J}_\pm = (J_{\pm,t}) \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$, and choose a smooth family $(\mathbf{J}_s = (J_{s,t}))_{s\in\mathbb{R}} \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$ such that there exists a compact set $K \subseteq X$ such that $J_{s,t} = J_{\mathrm{gb}}$ on $X\backslash K$ for all $(s,t) \in \mathbb{R}\times[0,1]$, and such that $\mathbf{J}_s = \mathbf{J}_-$ for $s \ll 0$ and $\mathbf{J}_s = \mathbf{J}_+$ for $s \gg 0$.*

*There exists a constant $\kappa > 0$ such that if $\|\partial_s H_s\|_{L^\infty} < \kappa$ then the conclusion of the previous theorem holds. That is, if $\|\partial_s H_s\|_{L^\infty} < \kappa$ then for any sequence $(u_m = (x_m,\tau_m))_{m\in\mathbb{N}} \subseteq \mathscr{M}^\alpha(H_s,\mathbf{J}_s)_a^b$, and any sequence $(s_m) \subseteq \mathbb{R}$, the reparametrized sequence $u_m(\cdot + s_m)$ has a subsequence which converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times[0,1],X) \times C^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$.*

**2.8** Remark. *We remark that because we are assuming that all our Hamiltonians are constant outside of a compact set, the only thing one needs to prove in the above two theorems is that the Lagrange multiplier component $\tau$ of a flow line $u = (x,\tau)$ is uniformly bounded. The bound on the loop component $x$ comes essentially "for free" from our assumption that the almost complex structures we work with are all geometrically bounded outside of a compact set; see for instance [21]. Later on we will need to work with Hamiltonians that are* **not** *constant outside a compact set; hence more work will need to be done here (cf. the discussion in Section 3.2).*

**2.9** Remark. *It follows from Theorem 2.6 that given $-\infty < a < b < \infty$, the subset $\mathrm{Crit}(\mathscr{A}_H)_a^b$ is compact (by the Arzelà-Ascoli theorem). Thus as $\mathscr{A}_H$ is Morse [resp. Morse-Bott if $\alpha \neq 0$], the set $\mathrm{Crit}(\mathscr{A}_H)_a^b$ is at most finite [resp. has at most finitely many components].*

## 2.4 The definition of $\mathrm{RFH}^\alpha_*(H)$

Assume $(\Sigma,L,\alpha)$ is a non-degenerate Rabinowitz admissible triple. In particular $\Sigma$ is transverse to $L$. The Rabinowitz action functional $\mathscr{A}_H$ is Morse-Bott by Lemma 2.5.2. The aim of this subsection is to "do Floer homology" with $\mathscr{A}_H$. One way to take into account the fact that we are in a Morse-Bott situation is to study the moduli spaces of **gradient flow lines with cascades**, as introduced originally by Frauenfelder in [26]. Actually if $\alpha \neq 0$ then the Rabinowitz action

functional is Morse, and in this case one can just work with normal gradient flow lines. The reader should bear in mind therefore that a lot of what follows can be considerably simplified when $\alpha \neq 0$.

Let $h : \mathrm{Crit}(\mathscr{A}_H) \to \mathbb{R}$ denote a Morse function and a fix Riemannian metric $g_h$ on $\mathrm{Crit}(\mathscr{A}_H)$ such that the flow $\phi_t^h : \mathrm{Crit}(\mathscr{A}_H) \to \mathrm{Crit}(\mathscr{A}_H)$ of $-\nabla h := -\nabla_{g_h} h$ is Morse-Smale. We abbreviate

$$C(h) := \mathrm{Crit}(h) \subseteq \mathrm{Crit}(\mathscr{A}_H)$$

$$C(h)_a^b := \mathrm{Crit}(h) \cap \mathrm{Crit}(\mathscr{A}_H)_a^b \quad .$$

Note that $C(h)_a^b$ is finite (cf. Remark 2.9).

Fix $\mathbf{J} = (J_t) \subseteq \mathscr{J}_{\mathrm{gb}}(X, \omega; J_{\mathrm{gb}})$.

**2.10** Definition. *Fix $m \in \mathbb{N}$. Given $\zeta_-, \zeta_+ \in C(h)_a^b$, a* **flow line from $\zeta_-$ to $\zeta_+$ with $m$ cascades** *is an $m$-tuple $(\mathbf{u}, \mathbf{T}) = ((u_i)_{i=1,\dots,m}, (T_i)_{i=1,\dots,m-1})$ of gradient flow lines . $u_i : \mathbb{R} \to P_\alpha(X, L) \times \mathbb{R}$ of $(H, \mathbf{J})$ and real numbers $T_i \geq 0$ such that*

$$\lim_{s \to -\infty} u_1(s) \in W^u(\zeta_-; -\nabla h), \qquad \lim_{s \to +\infty} u_m(s) \in W^s(\zeta_+; -\nabla h),$$

$$\lim_{s \to -\infty} u_{i+1}(s) = \phi_{T_i}^h \left( \lim_{s \to \infty} u_i(s) \right) \quad \text{for } i = 1, \dots, m-1.$$

*We denote the space of flow lines with $m$ cascades from $\zeta_-$ to $\zeta_+$ by $\widetilde{\mathscr{M}}_m(\zeta_-, \zeta_+)$, and we denote by $\mathscr{M}_m(\zeta_-, \zeta_+)$ the quotient $\widetilde{\mathscr{M}}_m(\zeta_-, \zeta_+)/\mathbb{R}^m$, where $\mathbb{R}^m$ acts by time shift on each of the $m$ cascades. We define a flow line with $0$ cascades to simply be a gradient flow line of $-\nabla h$, and denote by $\widetilde{\mathscr{M}}_0(\zeta_-, \zeta_+)$ the set of flow lines with $0$ cascades that are asymptotically equal to $\zeta_\pm$. We put $\mathscr{M}_0(\zeta_-, \zeta_+) := \widetilde{\mathscr{M}}_0(\zeta_-, \zeta_+)/\mathbb{R}$. Finally we define*

$$\mathscr{M}(\zeta_-, \zeta_+) := \bigcup_{m \in \mathbb{N} \cup \{0\}} \mathscr{M}_m(\zeta_-, \zeta_+).$$

**2.11** Definition. *Given a non-degenerate critical point $(x, \tau) \in \mathrm{Crit}(\mathscr{A}_H)$ with $\tau \neq 0$, set*

$$\mu(x, \tau) := \mu_{\mathsf{Ma}}(x, \tau) - \frac{1}{2}\chi(x, \tau),$$

*where $\mu_{\mathsf{Ma}}(x, \tau)$ is the* **Maslov index** *of the path $\zeta(t) := x(t/\tau)$ (see [34, Section 5.5] for the precise sign conventions we are using). If $\tau = 0$, set*

$$\mu(\underline{x}, 0) : -\frac{n-1}{2}.$$

*If $\zeta \in C(h)$ define $\mu_h(\zeta) := \mu(\zeta) + i_h(\zeta)$, where $i_h(\zeta)$ is the Morse index of $\zeta$ as a critical point of $h$ (thus $i_h(x, \tau) = 0$ whenever $\tau \neq 0$). Our sign conventions imply that for all $\zeta \in C(h)$,*

$$\mu_h(\zeta) \in \begin{cases} \mathbb{Z}, & \text{if } n \text{ is odd,} \\ \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}, & \text{if } n \text{ is even.} \end{cases}$$

The following theorem is part of the standard Floer homology package, the key ingredient being Theorem 2.6. The index computation is probably the most non-routine element - full details of this aspect can be found in [34].

**2.12** Theorem. *There exists an integer valued function $\mu_h : C(h) \to \mathbb{Z}$ with the following property: for a generic choice of $\mathbf{J}$ and a generic Morse-Smale metric $g_h$ on $\mathrm{Crit}(\mathscr{A}_H)$ the moduli spaces $\mathscr{M}(\zeta_-, \zeta_+)$ for $\zeta_\pm \in C(h)$ are smooth manifolds of finite dimension*

$$\dim \mathscr{M}(\zeta_-, \zeta_+) = \mu_h(\zeta_-) - \mu_h(\zeta_+) - 1.$$

*Moreover if $\mu_h(\zeta_-) = \mu_h(\zeta_+) + 1$ then $\mathscr{M}(\zeta_-, \zeta_+)$ is compact, and hence a finite set.*

Denote by

$$\mathrm{CRF}^{\alpha}_{*}(H,h)^b_a := C_*(h)^b_a \otimes \mathbb{Z}_2,$$

where the grading $*$ is given by the function $\mu_h$ from Definition 2.11.

Given $\zeta_-, \zeta_+ \in C(h)^b_a$ with $\mu_h(\zeta_-) = \mu_h(\zeta_+) + 1$, we define the number $n(\zeta_-, \zeta_+) \in \mathbb{Z}_2$ to be the parity of the finite set $\mathscr{M}(\zeta_-, \zeta_+)$. If $\zeta_+ \in C(h)^b_a$ has $\mu_h(\zeta_-) \neq \mu_h(\zeta_+) + 1$, set $n(\zeta_-, \zeta_+) = 0$. Now we can define the boundary operator

$$\partial^b_a = \partial^b_a(H, \mathbf{J}, h, g_h) : \mathrm{CRF}^{\alpha}_{*}(H,h)^b_a \to \mathrm{CRF}^{\alpha}_{*-1}(H,h)^b_a$$

as the linear extension of

$$\partial^b_a \zeta_- := \sum_{\zeta_+ \in C(h)^b_a} n(\zeta_-, \zeta_+)\zeta_+.$$

It follows directly from Theorem 2.12 that $\partial^b_a \circ \partial^b_a = 0$, and thus $\{\mathrm{CRF}^{\alpha}_{*}(H,h)^b_a, \partial^b_a\}$ carries the structure of a differential $\mathbb{Z}_2$-vector space. We denote by $\mathrm{RFH}^{\alpha}_{*}(H, \mathbf{J}, h, g_h)^b_a$ its homology. The standard theory of continuation homomorphisms in Floer theory show that the homology $\mathrm{RFH}^{\alpha}_{*}(H, \mathbf{J}, h, g_h)^b_a$ is independent up to canonical isomorphism of the choices of $h$, $\mathbf{J}$, and $g_h$, and thus we omit them from the notation and write simply $\mathrm{RFH}^{\alpha}_{*}(H)^b_a$.

Suppose $-\infty < a < a' < b < \infty$. Let $\mathrm{p}^b_{a,a'} : \mathrm{CRF}^{\alpha}_{*}(H,h)^b_a \to \mathrm{CRF}^{\alpha}_{*}(H,h)^b_{a'}$ denote the projection along $\mathrm{CRF}^{\alpha}_{*}(H,h)^{a'}_a$. Since the action decreases along gradient flow lines, $\mathrm{p}^b_{a,a'}$ commutes with the boundary operators $\partial^b_a$ and $\partial^b_{a'}$, and hence induces a map

$$\mathrm{p}^b_{a,a'} : \mathrm{RFH}^{\alpha}_{*}(H)^b_a \to \mathrm{RFH}^{\alpha}_{*}(H)^b_{a'}.$$

Similarly given $-\infty < a < b < b' < \infty$ the inclusion $C(h)^b_a \hookrightarrow C(h)^{b'}_a$ induces maps

$$\mathtt{i}^{b,b'}_a : \mathrm{RFH}^{\alpha}_{*}(H)^b_a \to \mathrm{RFH}^{\alpha}_{*}(H)^{b'}_a.$$

The complexes $\{\mathrm{RFH}^{\alpha}_{*}(H)^b_a, \mathtt{p}, \mathtt{i}\}$ form a bidirect system of $\mathbb{Z}_2$-vector spaces, and hence we can define

$$\mathrm{RFH}^{\alpha}_{*}(H) := \varinjlim_{a \downarrow -\infty} \varprojlim_{b \uparrow \infty} \mathrm{RFH}^{\alpha}_{*}(H)^b_a.$$

In fact, if $(\Sigma_{\pm}, L, \alpha)$ are both non-degenerate Rabinowitz admissible pairs, and $H_{\pm} \in \mathscr{D}_0(\Sigma_{\pm})$ are defining Hamiltonians with the property that there exists a Rabinowitz admissible virtually contact homotopy $(H_s, \lambda_s)_{s \in \mathbb{R}}$ with $(H_s, \lambda_s) = (H_-, \lambda_-)$ for $s \leq 0$ and $(H_s, \lambda_s) = (H_+, \lambda_+)$ for $s \geq 0$ then one can prove

$$\mathrm{RFH}^{\alpha}_{*}(H_-) \cong \mathrm{RFH}^{\alpha}_{*}(H_+).$$

This is a standard Floer theoretical argument, the key ingredient being Theorem 2.7. Details can be found in [34] if the reader is unconvinced.

It follows that if $H \in \mathscr{D}_0(\Sigma)$ then $\mathrm{RFH}^{\alpha}_{*}(H)$ depends only on $\Sigma, L, X$ and $\alpha$. Thus we can finally make the following definition:

**2.13** Definition. *If $(\Sigma, L, \alpha)$ is a non-degenerate Rabinowitz admissible triple, we define the* **Lagrangian Rabinowitz Floer homology** *of $(\Sigma, L, X, \alpha)$ by*

$$\mathrm{RFH}^{\alpha}_{*}(\Sigma, L, X) := \mathrm{RFH}^{\alpha}_{*}(H) \quad \text{for any } H \in \mathscr{D}_0(\Sigma).$$

Moreover, since $\mathrm{RFH}^{\alpha}_{*}(H)$ is invariant under isotopies of $\Sigma$ through Rabinowitz admissible virtually restricted contact type homotopies we can even define $\mathrm{RFH}^{\alpha}_{*}(\Sigma, L, X)$ even when $(\Sigma, L, \alpha)$ is not non-degenerate, simply by first isotopying $\Sigma$ through Rabinowitz admissible virtually restricted contact type homotopies to a new hypersurface $\Sigma'$ such that $(\Sigma', L, \alpha)$ is non-degenerate (such a hypersurface $\Sigma'$ exists by Lemma 2.5.3), and then **defining**

$$\mathrm{RFH}^{\alpha}_{*}(\Sigma, L, X) := \mathrm{RFH}^{\alpha}_{*}(\Sigma', L, X).$$

We have now completed the proof of Statement (1) of Theorem A. The proof of Statement (2) is essentially immediate, since if there were no characteristic chords representing $0 \in \Pi_L$ then the Rabinowitz Floer complex $\mathrm{CRF}^0_*(H,h)$ would reduce to the Morse complex of the Morse function $h : \Sigma \cap L \to \mathbb{R}$, and thus the Rabinowitz Floer homology would agree with the Morse homology of $h$, and hence the singular homology of $\Sigma \cap L$. This is non-zero.

We will discuss the remaining statements of Theorem A (on leaf-wise intersections) next.

## 2.5 Leaf-wise intersection points

In this section we address Statements (3) and (4) of Theorem A. The material in this section is essentially all taken from [10]. Assume throughout this section that $(\Sigma, L, \alpha)$ is a non-degenerate Rabinowitz admissible triple.

We need to study certain perturbations of the Rabinowitz action functional $\mathscr{A}_H$. We begin by defining a subset $\mathscr{X} \subseteq C^\infty(S^1, [0,\infty))$. In order to define $\mathscr{X}$, let us first associate to any element $\chi \in C^\infty(S^1, [0,\infty))$ the function $\bar{\chi} : [0,1] \to [0,\infty)$ defined by

$$\bar{\chi}(t) := \int_0^t \chi(\tau) d\tau.$$

Let $\mathscr{X} \subseteq C(S^1, [0,\infty))$ denote those functions $\chi$ whose associated function $\bar{\chi}$ satisfies the following conditions:

1. There exists $t_0 = t_0(\chi) \in (0,1]$ such that $\bar{\chi}(t) \equiv 1$ on $[t_0, 1]$;
2. On $[0, t_0]$ the function $\bar{\chi}$ is strictly increasing.

Note that the function $\chi = 1$ is an element of $\mathscr{X}$. It will sometimes be useful to restrict to the subset $\mathscr{X}_0 := \{\chi \in \mathscr{X} \,:\, t_0(\chi) < 1/2\}$.

**2.14** Remark. *Note that if $\chi \in \mathscr{X}$ then there is a unique function $\nu : [0,1) \to [0,t_0)$ such that*

$$\bar{\chi}(\nu(t)) = t \quad \textit{for all } t \in [0,1).$$

*One can extend $\nu$ to a continuous function $\nu : [0,1] \to [0,t_0]$ by setting $\nu(1) := t_0$.*

Given $H \in C^\infty(X, \mathbb{R})$ for which 0 is a regular value and $\chi \in \mathscr{X}$, we form a new **weakly time dependent** Hamiltonian $H^\chi : S^1 \times X \to \mathbb{R}$ by

$$H^\chi(t,x) := \chi(t) H(x).$$

It is easy to see that if $\mathscr{A}_{H^\chi}$ denotes the Rabinowitz action functional with $H$ replaced by $H^\chi$ then the critical points of $\mathscr{A}_{H^\chi}$ are the pairs $(x,\tau)$ with

$$\dot{x}(t) = \tau\chi(t) X_H(x), \quad x([0,1]) \subseteq H^{-1}(0).$$

Thus there is a natural bijective correspondence between $\mathrm{Crit}(\mathscr{A}_{H^\chi})$ and $\mathrm{Crit}(\mathscr{A}_H)$: if $(x,\tau) \in \mathrm{Crit}(\mathscr{A}_{H^\chi})$ then $(x \circ \nu, \tau) \in \mathrm{Crit}(\mathscr{A}_H)$, where $\nu : [0,1] \to [0,t_0]$ is as in Remark 2.14.

It is not hard to see that all of what we have accomplished so far goes through for the functions $H^\chi$. In particular, if $H \in \mathscr{D}_0(\Sigma)$ then the Rabinowitz Floer homology $\mathrm{RFH}^\alpha_*(H^\chi)^b_a$ is well defined, and moreover it is actually independent of the choice of $\chi \in \mathscr{X}$ and hence

$$\mathrm{RFH}^\alpha_*(H^\chi) = \mathrm{RFH}^\alpha_*(H^{\chi=1}),$$

which by definition is the Rabinowitz Floer homology $\mathrm{RFH}^\alpha_*(\Sigma, L, X)$. The details are carried out in [18, Section 3.2], and there are no differences in our case.

Let

$$\mathscr{F} := \left\{ F \in C^\infty_c(S^1 \times X, \mathbb{R}) \,:\, F(t,\cdot) \equiv 0 \text{ for all } t \in [0,1/2] \right\}.$$

If $\psi \in \mathrm{Ham}_c(X,\omega)$ then we can find $F \in \mathscr{F}$ such that $\psi = \phi_1^F$ ([10, Lemma 2.3]).

Given $F \in \mathscr{F}$ and $\chi \in \mathscr{X}$ the **perturbed Rabinowitz action functional** $\mathscr{A}^F_{H^\chi} : P_\alpha(X,L) \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathscr{A}^F_{H^\chi}(x,\tau) = \mathscr{A}_{H^\chi}(x,\tau) - \int_0^1 F(t,x(t))dt.$$

Denote by $\mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ the set of critical points of $\mathscr{A}^F_{H^\chi}$. Note that a pair $(x,\tau)$ belongs to $\mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ if and only if:

$$\dot{x} = \tau\chi(t)X_H(x) + X_F(t,x);$$
$$\int_0^1 \chi(t)H(x)dt = 0.$$

This perturbation is particularly interesting when $\chi \in \mathscr{X}_0$, since in this case the two functions $H^\chi$ and $F$ have disjoint time support, and hence if $(x,\tau) \in \mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ then $x$ is a Hamiltonian chord] of the time-dependent Hamiltonian $\tau H^\chi + F$, so that, $x(t) = \phi_t^{\tau H^\chi + F}(x(0))$. This leads to the following observation, which is proved in the same way as [10, Proposition 2.4], and explains why Lagrangian Rabinowitz Floer homology is useful in the study of Lagrangian leaf-wise intersection points.

**2.15** LEMMA. *Suppose $H \in \mathscr{D}(\Sigma)$ and $F \in \mathscr{F}$. Let $\psi = \phi_1^F$. Then for any $\chi \in \mathscr{X}_0$, there is a surjective map $e : \mathrm{Crit}(\mathscr{A}^F_{H^\chi}) \to \mathscr{L}W(\Sigma, L, \psi, \alpha)$ given by*

$$e(x,\tau) := x(0).$$

Given $(x,\tau) \in \mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ and $\mathbf{J} = (J_t) \subseteq \mathscr{J}(X,\omega)$ the linearization $\nabla^2_{\mathbf{J}}\mathscr{A}^F_{H^\chi}(x,\tau)$ of $\nabla_{\mathbf{J}}\mathscr{A}_H(x,\tau)$ at is given by

$$\nabla^2_{\mathbf{J}}\mathscr{A}^F_{H^\chi}(x,\tau)(\xi,h) := \begin{pmatrix} J_t(x)\nabla_t\xi + (\nabla_\xi J_t)\dot{x} - \tau\nabla_\xi\nabla_{J_t}H^\chi - h\nabla_{J_t}H^\chi - \nabla_\xi\nabla_{J_t}F \\ -\int_0^1 \mathrm{d}H^\chi(\xi)dt \end{pmatrix}.$$

**2.16** DEFINITION. *A critical point $(x,\tau) \in \mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ is called* **non-degenerate** *if $\nabla^2_{\mathbf{J}}\mathscr{A}^F_{H^\chi}(x,\tau)$ is surjective. An element $F \in \mathscr{F}$ is* **non-degenerate with respect to** *$(H^\chi,\alpha)$ if every critical point $(x,\tau) \in \mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ is non-degenerate. We denote by $\mathscr{F}_{\mathrm{reg}}(H^\chi, L, \alpha) \subseteq \mathscr{F}$ the set of non-degenerate functions with respect to $H^\chi$ and $\alpha$.*

The following result, which is the analogue of Lemma 2.5.2, is straightforward.

**2.17** LEMMA. *The perturbed Rabinowitz action functional $\mathscr{A}^F_{H^\chi}$ is Morse if every critical point $(x,\tau) \in \mathrm{Crit}(\mathscr{A}^F_{H^\chi})$ non-degenerate.*

The proof of the following result, which is the analogue of Lemma 2.5.3, is much less straightforward, but is very similar to [9, Appendix A].

**2.18** THEOREM. *The set $\mathscr{F}_{\mathrm{reg}}(H^\chi, L, \alpha)$ is generic in $\mathscr{F}$.*

Suppose now $(F_s)_{s\in\mathbb{R}} \subseteq \mathscr{F}$ is an asymptotically constant family, and $(\mathbf{J}_s = (J_{s,t}))_{s\in\mathbb{R}} \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$ is an asymptotically constant family of almost complex structures. As before, given $-\infty < a < b < \infty$, we denote by $\mathscr{M}^\alpha(H^\chi, F_s, \mathbf{J}_s)_a^b$ the moduli space of gradient flow lines $u$ satisfying the $s$-dependent equation

$$\partial_s u + \nabla_{\mathbf{J}_s}\mathscr{A}^{F_s}_{H^\chi}(u(s)) = 0,$$

which satisfy the asymptotic conditions

$$\lim_{s\to-\infty}\mathscr{A}^{F_s}_{H^\chi}(u(s)) \le b, \quad \lim_{s\to\infty}\mathscr{A}^{F_s}_{H^\chi}(u(s)) \ge a.$$

The following result is the analogue of Theorem 2.6, and is proved in a similar fashion. If the reader is unconvinced the details can be found in [9, Theorem 2.9].

**2.19** THEOREM. *Let $H \in \mathscr{D}_0(\Sigma)$, $\chi \in \mathscr{X}$ and $F_\pm \in \mathscr{F}$, and choose a smooth family $(F_s)_{s\in\mathbb{R}} \subseteq \mathscr{F}$ such that $F_s = F_-$ for $s \ll 0$ and $F_s = F_+$ for $s \gg 0$, and such that $F_s$ has compact support uniformly in $s$. Fix $\mathbf{J}_\pm = (J_{\pm,t}) \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$, and choose a smooth family $(\mathbf{J}_s = (J_{s,t}))_{s\in\mathbb{R}} \subseteq \mathscr{J}_{\mathrm{gb}}(X,\omega;J_{\mathrm{gb}})$ such that there exists a compact set $K \subseteq X$ such that $J_{s,t} = J_{\mathrm{gb}}$ on $X \backslash K$ for all $(s,t) \in \mathbb{R}\times[0,1]$, and such that $\mathbf{J}_s = \mathbf{J}_-$ for $s \ll 0$ and $\mathbf{J}_s = \mathbf{J}_+$ for $s \gg 0$. Fix $-\infty < a < b < \infty$ and suppose $(u_m = (x_m,\tau_m))_{m\in\mathbb{N}} \subseteq \mathscr{M}^\alpha(H^\chi, F_s, \mathbf{J}_s)_a^b$. Then for any sequence $(s_m) \subseteq \mathbb{R}$, the reparametrized sequence $u_m(\cdot + s_m)$ has a subsequence which converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times[0,1],X)\times C^\infty_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$.*

Suppose $F \in \mathscr{F}_{\mathrm{reg}}(H^\chi,\alpha)$ [resp. $F \in \mathscr{F}_{\mathrm{reg}}(H^\chi,L,\alpha)$]. Then one can define the Rabinowitz Floer homology $\mathrm{RFH}^\alpha_*(H^\chi,F)$. This is defined in exactly the same way as before, only since we are now in the Morse situation, no additional Morse function $h$ is needed. Moreover, by choosing an $s$-dependent homotopy $F_s$ from $F$ to 0, and taking note of Theorem 2.19, one sees that the usual continuation homomorphisms are well-defined and isomorphisms. Thus we conclude:

$$\mathrm{RFH}^\alpha_*(H^\chi,F) \cong \mathrm{RFH}^\alpha_*(H^\chi,0) = \mathrm{RFH}^\alpha_*(H)$$

(see [9, Section 2.3] for more information).

In particular, if one can find $F \in \mathscr{F}$ such that $\mathrm{Crit}(\mathscr{A}^F_{H^\chi}) = \emptyset$ then $\mathscr{A}^F_{H^\chi}$ is trivially Morse, and $\mathrm{RFH}^\alpha_*(H^\chi,F) = 0$. This observation, combined with Lemma 2.15, completes the proof of Statement (3) of Theorem A. The proof of Statement (4) also follows, by making use of Remark 1.6.

# 3 Twisted cotangent bundles and Mañé supercritical hypersurfaces

We now go back to the setting described on page 5 of twisted cotangent bundles.

## 3.1 The Mañé critical value

We now recall the definition of the critical value $c(H,\sigma)$, as introduced by Mañé in [31], which play a decisive role in all that follows. General references for the results stated below are [23, Proposition 2-1.1] or [17, Appendix A]. We will then explain how to modify the definition of the critical value $c(H,\sigma)$ to take into account a given $\pi_1$-injective submanifold $S \subseteq M$ for which $\sigma|_S = 0$. This leads to a new critical values $c(H,\sigma,S)$.

Fix an autonomous Tonelli Hamiltonian $H \in C^\infty(X,\mathbb{R})$, and denote by $\widetilde{H} \in C^\infty(\widetilde{X},\mathbb{R})$ the lift of $H$ to the universal cover $\widetilde{X}$. We define the the **Mañé critical value** associated to $H$ and $\sigma$ by

$$c(H,\sigma) := \inf_\theta \sup_{q\in\widetilde{M}} \widetilde{H}(q,-\theta_q), \tag{3.1}$$

where the infimum is taken over all primitives $\theta$ of $\widetilde{\sigma}$. Since $H$ is superlinear, $c(H,\sigma) < \infty$ if and only if $\widetilde{\sigma}$ admits a bounded primitive.

**3.1** REMARK. *The strange looking sign convention in* (3.1) *is due to the fact that we are using the "unnatural" sign convention that the canonical symplectic form on $X$ is given by $d\lambda_0$ (rather than $-d\lambda_0$).*

If we only look at primitives $\theta$ of $\widetilde{\sigma}$ that belong to $\Omega^1_S(\widetilde{\sigma})$ then we obtain a new critical value

$$c(H,\sigma,S) := \inf_{\theta\in\Omega^1_S(\widetilde{\sigma})} \sup_{q\in\widetilde{M}} \widetilde{H}(q,-\theta_q),$$

where if $\Omega^1_S(\widetilde{\sigma}) = \emptyset$ we define $c(H,\sigma,S) := \infty$.

Clearly

$$c(H,\sigma) \leq c(H,\sigma,S),$$

and $c(H,\sigma,S)$ only depends on $S$ through the set $\Omega^1_S(\widetilde{\sigma})$. In particular if $\pi_1(S)$ is finite then $c(H,\sigma)=c(H,\sigma,S)$, as in this case $\Omega^1_S(\widetilde{\sigma})$ denotes the set of **all** primitives of $\sigma$.

**3.2** DEFINITION. *A closed connected hypersurface $\Sigma\subseteq X$ is called a* **Tonelli hypersurface** *if there exists a Tonelli Hamiltonian $H\in C^\infty(X,\mathbb{R})$ for which $0$ is a regular value of $H$ with $\Sigma=H^{-1}(0)$. In this case we say that $H$ is a* **defining Tonelli Hamiltonian** *for $\Sigma$. Note that the condition that $\Sigma$ is a Tonelli hypersurface does not depend on the choice of Riemannian metric on $M$.*

Recall that $\Omega^1_\Sigma(\widetilde{\sigma})$ denotes the set of primitives $\theta$ of $\widetilde{\sigma}$ for which $\lambda_\theta:=\widetilde{\lambda}_0+\widetilde{\pi}^*\theta$ is a primitive of $\omega_{\widetilde{\sigma}}$ making $\Sigma$ into a hypersurface of virtual restricted contact type (cf. Definition 1.1).

**3.3** LEMMA. *Let $\Sigma$ denote a Tonelli hypersurface. Then $\Omega^1_\Sigma(\widetilde{\sigma})\neq\emptyset$ if there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ satisfying $c(H,\sigma)<0$. Similarly $\Omega^1_\Sigma(\widetilde{\sigma})\cap\Omega^1_S(\widetilde{\sigma})\neq\emptyset$ if there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ satisfying $c(H,\sigma,S)<0$.*

**Proof.** We prove the second statement only (the first is entirely analogous). Suppose $H$ is a defining Tonelli Hamiltonian for $\Sigma$ satisfying $c(H,\sigma,S)<0$. By the definition (3.1) of $c(H,\sigma,S)$ there exists $\varepsilon>0$ and a bounded primitive $\theta\in\Omega^1_S(\widetilde{\sigma})$ of $\widetilde{\sigma}$ such that the lift $\widetilde{H}$ of $H$ satisfies $\widetilde{H}(q,-\theta_q)<-\varepsilon$ for all $q\in\widetilde{M}$. We show that $\theta\in\Omega^1_\Sigma(\widetilde{\sigma})$. Set $\lambda_\theta:=\widetilde{\lambda}_0+\widetilde{\pi}^*\theta$. Since $\theta$ is bounded, we need only check that

$$\inf_{(q,p)\in\widetilde{\Sigma}}\lambda_\theta(X_{\widetilde{H}}(q,p))>0,$$

where $X_{\widetilde{H}}$ is the symplectic gradient of $\widetilde{H}$ with respect to the lifted symplectic form $\widetilde{\omega}_\sigma:=d\widetilde{\lambda}_0+\widetilde{\pi}^*\widetilde{\sigma}$.

Fix $(q,p)\in\widetilde{\Sigma}$, and let

$$f(s):=\widetilde{H}(q,(1-s)\theta_q+sp).$$

A simple computation yields

$$\lambda_\theta(X_{\widetilde{H}}(q,p))=f'(1).$$

Now note that $f(0)<-\varepsilon$ and $f(1)=0$, and since $H$ is Tonelli, $f$ is convex and thus we must have $f'(1)>\varepsilon$. ∎

We can now give the following key definition.

**3.4** DEFINITION. *We say a Tonelli hypersurface is a* **Mañé supercritical** *hypersurface there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ satisfying $c(H,\sigma)<0$. Note that such hypersurfaces can only exist when $\widetilde{\sigma}$ admits a bounded primitive.*

*We say that a pair $(\Sigma,S)$ consisting of Tonelli hypersurface $\Sigma$ and a closed connected $\pi_1$-injective submanifold $S\subseteq M$ for which $\sigma|_S=0$ are a* **Mañé supercritical pair** *if there exists a defining Tonelli Hamiltonian $H$ for $\Sigma$ satisfying $c(H,\sigma,S)<0$. Note that if $(\Sigma,S)$ is a Mañé supercritical pair then $\Sigma$ is necessarily itself a Mañé supercritical hypersurface.*

**3.5** EXAMPLE. *Here is an example (due to Alberto Abbondandolo) that illustrates the difference between simply asking that $\Sigma$ is a Mañé supercritical hypersurface, and asking that a pair $(\Sigma,S)$ forms a Mañé supercritical pair. Take $M=\mathbb{T}^n$ and $\sigma=0$, and take $S=S^1\times\{\mathrm{pt}\}$. Define $H:X\to\mathbb{R}$ by*

$$H(q,p):=\frac{1}{2}\left|p-dq^1\right|^2.$$

*One easily sees that*

$$c(H,\sigma)=0,$$

*but that*

$$c(H,\sigma,S)=1/2.$$

*Thus $H^{-1}(k)$ is a Mañé supercritical hypersurface provided $k > 0$, whereas the pair $(H^{-1}(k), S)$ is a Mañé supercritical pair only when $k > 1/2$.*

*In fact, $H^{-1}(k) \cap N^*S = \emptyset$ if $k < 1/2$. For $k > 1/2$, not only is $H^{-1}(k) \cap N^*S$ non-empty, but by Corollary C the hypersurface $H^{-1}(k)$ can never be displaced from $N^*S$ by an element of $\mathrm{Ham}_c(X, d\lambda_0)$.*

**3.6** Remark. *It is also possible to define the critical value $c(H, \sigma, S)$ in the Lagrangian framework. More precisely, if $L$ is a Tonelli Lagrangian then one can define a critical value $c(L, \sigma, S)$ which encodes the same information as $c(H, \sigma, S)$, in the sense that if $H$ and $L$ are Fenchel dual to each other then $c(H, \sigma, S) = c(L, \sigma, S)$. See [34, Section 8.5] for more information.*

## 3.2 Lagrangian Rabinowitz Floer homology with Tonelli Hamiltonians

Note that a defining Tonelli Hamiltonian $H$ for a Mañé supercritical pair $(\Sigma, S)$ is in particular an element of $\mathscr{D}(\Sigma)$, but that $H$ is **not** constant outside a compact set, and hence $H \notin \mathscr{D}_0(\Sigma)$. Thus it is not a priori clear that one can use $H$ to define the Rabinowitz Floer homology of $\Sigma$, and even if we could, whether it would yield the same Rabinowitz Floer homology as the one developed in Section 2.4. The key difficulty here is that since $H$ is not compactly supported, the compactness results from Theorem 2.6 and Theorem 2.7 fail (see Remark 2.8).

In [3] Abbondandolo and Schwarz showed how such compactness could still be obtained (in the setting of "standard" Floer homology on cotangent bundles equipped with standard symplectic form $d\lambda_0$) for a wide class of Hamiltonians. Roughly speaking, they prove $L^\infty$ estimates for Hamiltonians that, outside of a compact set, are **quadratic** in the fibres (see [3, Section 1.5] for the precise definition). Their idea is based upon isometrically embedding $X$ into $\mathbb{R}^{2N}$ (via Nash's theorem), and combining Calderon-Zygmund estimates for the Cauchy-Riemann operator with certain interpolation inequalities. We remark that in order for these $L^\infty$ estimates to hold it is important that the almost complex structure we choose lies sufficiently close (in the $L^\infty$ norm) to the **metric almost complex structure** $J_g$ associated to some fixed Riemannian metric $g = \langle \cdot, \cdot \rangle$ on $M$. This is the unique almost complex structure on $X$ with the property that under the splitting $TX \cong TM \oplus T^*M$ determined by the metric (see Section 3.5 below), $J_g$ acts as

$$J_g = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

A Tonelli Hamiltonian $H \in C^\infty(X, \mathbb{R})$ is **electromagnetic at infinity** (with respect to $g$) if there exists a smooth positive function $a \in C^\infty(M, \mathbb{R}^+)$, a smooth 1-form $\beta \in \Omega^1(M)$, a smooth function $V \in C^\infty(M, \mathbb{R})$, and a real number $R > 0$ such that

$$H(q,p) = \frac{1}{2} a(q) \left| p - \beta_q \right|^2 + V(q) \quad \text{for all } (q,p) \in X \text{ with } |p| \geq R.$$

In [34] we use a version of the argument of Abbondandolo and Schwarz mentioned above to show that the Lagrangian Rabinowitz Floer homology $\mathrm{RFH}_*^\alpha(H)$ is well defined when $H$ is a Tonelli Hamiltonian which is electromagnetic at infinity and satisfies $c(H, \sigma, S) < 0$, and moreover that this Rabinowitz Floer homology is the same as the one defined using Hamiltonians which are constant outside a compact set. Actually strictly speaking in order for this result to hold, one may need to rescale $\sigma$ (this is so $\omega_\sigma$-compatible almost complex structures that are sufficiently close in the $L^\infty$-norm to the metric almost complex structure $J_g$ exist); this does not actually entail any loss of generality, as the Lagrangian Rabinowitz Floer homology of Section 2.4 is invariant under such rescaling. As such we will ignore this subtlety throughout. See [34, Lemma 8.12].

The following result is a minor variant of [24, Corollary 20].

**3.7** Proposition. *Suppose $\Sigma = H^{-1}(0)$ is a regular energy value of a Tonelli Hamiltonian $H \in C^\infty(X, \mathbb{R})$ with $c(H, \sigma, S) < 0$. Then there exists another Tonelli Hamiltonian $\overline{H}$ that is electromagnetic at infinity and satisfies:*

$$H \equiv \overline{H} \quad \text{on } \{H \leq 1\};$$

$$c(H,\sigma,S)=c(\overline{H},\sigma,S).$$

## 3.3 Hypotheses

From now on we fix a Tonelli hypersurface $\Sigma$ and a closed connected $\pi_1$-injective submanifold $S\subseteq M$ of dimension $0\le d\le n$ such that $\sigma|_S=0$, and such that $(\Sigma,S)$ form a Mañé supercritical pair. Without loss of generality (as far as the Lagrangian Rabinowitz Floer homology is concerned), we can and will assume that $(\Sigma,N^*S,\alpha)$ is non-degenerate for every $\alpha\in\Pi_S$. In particular $\Sigma$ is transverse to $N^*S$.

Proposition 3.7 implies that we may choose a Tonelli Hamiltonian $H\in C^\infty(X,\mathbb{R})$ that is electromagnetic at infinity and satisfies $\Sigma=H^{-1}(0)$ with $c(H,\sigma,S)<0$, and thus we may compute the Lagrangian Rabinowitz Floer homology $\mathrm{RFH}^\alpha_*(\Sigma,N^*S,X)$ using $H$:

$$\mathrm{RFH}^\alpha_*(H)\cong\mathrm{RFH}^\alpha_*(\Sigma,N^*S,X).$$

The aim of the rest of this paper is to compute $\mathrm{RFH}^\alpha_*(H)$.

## 3.4 Grading

Before getting started on computing $\mathrm{RFH}^\alpha_*(H)$, we will spend a little time discussing the grading on the Lagrangian Rabinowitz Floer homology in the specialized situation we are working in now. In fact, there is a particularly satisfying solution to the grading issue on twisted cotangent bundles. This is because every twisted cotangent bundle possesses a **Lagrangian distribution**, namely the vertical distribution $T^{\mathsf{v}}X$ (i.e. the tangent spaces to the fibres: $T^{\mathsf{v}}_{(q,p)}X:=T_{(q,p)}T^*_qM$). The vertical distribution singles out a distinguished class of symplectic trivializations - those that are **vertical preserving**. Namely, if $x\in P(X,N^*S)$, a trivialization $\Phi:[0,1]\times\mathbb{R}^{2n}\to x^*TX$ is called **vertical preserving** if

$$\Phi(t,V_0)=T^{\mathsf{v}}_{x(t)}X\quad\text{for all }t\in[0,1],$$

where $V_0:=\{0\}\times\mathbb{R}^n$ is the vertical subspace. Such trivializations always exist (cf. [3, Lemma 1.2]).

Given a critical point $(x,\tau)$, let $\Phi:[0,1]\times\mathbb{R}^{2n}\to x^*TX$ denote a vertical preserving trivialization, and define a path $\vartheta:[0,1]\to\mathsf{Lag}(\mathbb{R}^{2n},\omega_0)$ by

$$\Phi(t,\vartheta(t))=\mathrm{d}\phi^H_{\tau t}(x(0))(T^{\mathsf{v}}_{x(0)}X).$$

Now define

$$\mu_{\mathsf{Ma}}(x,\tau):=\mu_{\mathsf{RS}}(\vartheta,V_0),$$

where $\mu_{\mathsf{RS}}$ is the **Robbin-Salamon index** [39] (although be warned - our sign convention for $\mu_{\mathsf{RS}}$ matches [2] rather than [39]). This index $\mu_{\mathsf{Ma}}(x,\tau)$ is independent of the vertical preserving trivialization $\Phi$ (cf. [3, Lemma 1.3.(ii)]).

In fact it will also be convenient to introduce a grading shift of $d-\frac{n-1}{2}$ (recall $d=\dim S$). This choice is motivated by Theorem 3.13 below, and it also ensures our grading is always $\mathbb{Z}$-valued. Thus in this section we define

$$\mu(x,\tau):=\begin{cases}\mu_{\mathsf{Ma}}(x,\tau)-\frac{1}{2}\chi(x,\tau)+d-\frac{n-1}{2}, & \tau\neq0,\\ d-n+1, & \tau=0.\end{cases}$$

## 3.5 The free time action functional

Let us abbreviate $W:=TM$. Denote again by $\pi$ the footpoint map $W\to M$. Let us fix once and for all an auxiliary Riemannian metric $g$ on $M$. The Riemannian metric $g$ defines a **horizontal-vertical** splitting of $TW$: given $w=(q,v)\in W$ we write

$$T_wW=T^{\mathsf{h}}_wW\oplus T^{\mathsf{v}}_wW\cong T_qM\oplus T_qM;$$

here $T^{\mathsf{h}}_w W = \ker(\kappa_g : T_w W \to T_q M)$, where $\kappa_g$ is the connection map of the Levi-Civita connection $\nabla$ of $g$, and $T^{\mathsf{v}}_w W = \ker(\mathrm{d}\pi(w) : T_w W \to T_q M)$. Given $\xi \in TW$ we denote by $\xi^{\mathsf{h}}$ and $\xi^{\mathsf{v}}$ the horizontal and vertical components. The **Sasaki metric** $g_W$ on $W$ is defined by

$$g_W(\xi, \vartheta) := \left\langle \xi^{\mathsf{h}}, \vartheta^{\mathsf{h}} \right\rangle + \left\langle \xi^{\mathsf{v}}, \vartheta^{\mathsf{v}} \right\rangle .$$

Suppose $f \in C^\infty(W, \mathbb{R})$ is an arbitrary smooth function. Then $\mathrm{d}f(w) \in T^*_w W$, and thus its gradient $\nabla f(w) = \nabla_{g_W} f(w)$ lies in $T_w W$. Thus we can speak of the horizontal and vertical components

$$\nabla f^{\mathsf{h}}(w) := [\nabla f(w)]^{\mathsf{h}} \in T_q M;$$

$$\nabla f^{\mathsf{v}}(w) := [\nabla f(w)]^{\mathsf{v}} \in T_q M.$$

Let us go back to our fixed Hamiltonian $H$. The fact that $H$ is Tonelli implies there exists a unique Tonelli Lagrangian (that is, fibrewise strictly convex and superlinear) Lagrangian $L \in C^\infty(W, \mathbb{R})$ called the **Fenchel dual** Lagrangian to $H$, which is related to $H$ by:

$$H(q, p) := p(v) - L(q, v), \quad \text{where } \nabla L^{\mathsf{v}}(q, v) = p.$$

Since $H$ is electromagnetic at infinity, so is $L$ - that is, there exists a smooth positive function $a \in C^\infty(M, \mathbb{R}^+)$, a smooth 1-form $\beta \in \Omega^1(M)$, a smooth function $V \in C^\infty(M, \mathbb{R})$, and a real number $R > 0$

$$L(q, v) = \frac{1}{2} a(q) \left| v \right|^2 + \beta_q(v) - V(q) \quad \text{for all } (q, v) \in W \text{ with } |p| \geq R.$$

The aim of this section is to do Morse theory with a free time action functional $\mathscr{S}_L$ (defined below). Unfortunately the space $P(M, S)$ does not admit the structure of Hilbert manifold (it is only Fréchet manifolds, which is not sufficient for Morse theory). Thus throughout this section we will work with the $W^{1,2}$ Sobolev completion $\mathscr{P}(M, S)$ of $P(M, S)$.

Recall from the proof of Lemma 1.9 that **$\sigma$-area** $\Omega_\sigma : P(M, S) \to \mathbb{R}$ is defined by

$$\Omega_\sigma(q) := \int_{[0,1]\times[0,1]} \bar{q}^* \sigma,$$

where $q \in P_\alpha(M, S)$ and $\bar{q}$ is any filling of $q$ (i.e. any smooth map such that $\bar{q}(0, t) = q(t)$, $\bar{q}(1, t) = q_\alpha(t)$ and $\bar{q}([0, 1] \times \{0, 1\}) \subseteq S$). Let us note that $\Omega_\sigma$ extends to a functional on $\mathscr{P}(M, S)$ - if $q$ is of class $W^{1,2}$ then we choose the filling $\bar{q} \in W^{1,2}([0, 1] \times [0, 1], M) \cap C^0([0, 1] \times [0, 1], M)$.

We will study the **free time action functional**

$$\mathscr{S}_L : \mathscr{P}(M, S) \times \mathbb{R}^+ \to \mathbb{R}$$

which is defined by

$$\mathscr{S}_L(q, \tau) := \Omega_\sigma(q) + \tau \int_0^1 L\left(q, \frac{\dot{q}}{\tau}\right) dt.$$

In the case $\sigma = 0$, the functional $\mathscr{S}_L$ has been extensively studied in [24, 22].

The pair $(\sigma, g)$ defines a bundle endomorphism $Y = Y_{\sigma,g} \in \Gamma(\mathrm{End}(W))$ called the **Lorentz force** of $\sigma$ via:

$$\sigma_q(u, v) = \langle Y(q)u, v \rangle .$$

A standard computation tells us that if $(q, \tau) \in \mathscr{P}(M, S) \times \mathbb{R}^+$ and $(q_s, \tau_s)_{s\in(-\varepsilon,\varepsilon)} \subseteq \mathscr{P}(M, S) \times \mathbb{R}^+$ is a variation of $(q, \tau)$ with $\frac{\partial}{\partial s}\big|_{s=0} q_s(t) =: \eta(t)$ and $\frac{\partial}{\partial s}\big|_{s=0} \tau_s =: h$ then

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0} \mathscr{S}_L(q_s, \tau_s) = &\int_0^\tau \left\langle \nabla L^{\mathsf{h}}(\gamma, \dot{\gamma}) - \nabla_t(\nabla L^{\mathsf{v}}(\gamma, \dot{\gamma})) - Y(\gamma)\dot{\gamma}, v \right\rangle dt, \\ &- \frac{h}{\tau} \int_0^\tau E(\gamma, \dot{\gamma}) dt + [\langle \nabla L^{\mathsf{v}}(\gamma(1), \dot{\gamma}(1), v(1) \rangle - \langle \nabla L^{\mathsf{v}}(\gamma(0), \dot{\gamma}(0), v(0) \rangle]\end{aligned}$$

Thus $\frac{\partial}{\partial s}\big|_{s=0}\mathscr{S}_L(q_s,\tau_s)=0$ for all such variations $(q_s,\tau_s)$ if and only if $\gamma(t):=q(t/\tau)$ satisfies the **Euler-Lagrange equations**

$$\nabla L^{\mathsf{h}}(\gamma,\dot\gamma)-\nabla_t(\nabla L^{\mathsf{v}}(\gamma,\dot\gamma))-Y(\gamma)\dot\gamma=0 \tag{3.2}$$

together with the **energy constraint**

$$\int_0^1 E(\gamma,\dot\gamma)dt=0, \tag{3.3}$$

and the boundary conditions

$$\langle\nabla L^{\mathsf{v}}(\gamma(1),\dot\gamma(1),u\rangle=\langle\nabla L^{\mathsf{v}}(\gamma(0),\dot\gamma(0),v\rangle \quad \text{for all } u\in T_{\gamma(0)}S \text{ and } v\in T_{\gamma(1)}S. \tag{3.4}$$

Since $L$ is electromagnetic at infinity, $\mathscr{S}_L$ is of class $C^{1,1}$ on $\mathscr{P}(M,S)\times\mathbb{R}^+$ (see [4, 5]).

It will be useful to consider the **fixed period action functional**. Given $\tau\in\mathbb{R}^+$ let us denote by $\mathscr{S}_L^\tau$ the functional defined by

$$\mathscr{S}_L^\tau(q):=\mathscr{S}_L(q,\tau).$$

Note that

$$\mathrm{d}\mathscr{S}_L^\tau(q)(\eta)=\mathrm{d}\mathscr{S}_L(q,\tau)(\eta,0).$$

Thus if $(q,\tau)\in\mathrm{Crit}(\mathscr{S}_L)$ then $q\in\mathrm{Crit}(\mathscr{S}_L^\tau)$.

By definition, the **Morse index** $m(q,\tau)$ of a critical point $(q,\tau)\in\mathrm{Crit}(\mathscr{S}_L)$ is the maximal dimension of a subspace $W^{1,2}(q^*TW)\times\mathbb{R}$ on which the Hessian $\nabla^2\mathscr{S}_L(q,\tau)$ of $\mathscr{S}_L$ at $(q,\tau)$ is negative definite. Similarly we denote by $m_\tau(q)$ the Morse index of a critical point $q\in\mathrm{Crit}(\mathscr{S}_L^\tau)$, that is, the maximal dimension of a subspace of $W^{1,2}(q^*W)$ on which the Hessian $\nabla^2\mathscr{S}_L^\tau$ of $\mathscr{S}_L^\tau$ is negative definite. It is well known that for Tonelli Lagrangians the Morse index $m_\tau(q)$ is always finite [25, Section 1].

We define the **nullity** $n(q,\tau)$ of a critical point of $\mathscr{S}_L$ to be

$$n(q,\tau):=\dim\ker(\nabla_q^2\mathscr{S}_L^\tau),$$

and we say that a critical point $(q,\tau)\in\mathscr{P}(M,S)\times\mathbb{R}^+$ is **non-degenerate** if $n(q,\tau)=0$. Since we have assumed that our fixed Hamiltonian $H$ is non-degenerate, it actually follows that every critical point of $\mathscr{S}_L$ is non-degenerate. This is because there is a simple relationship between the critical points of $\mathscr{S}_L$ and those of $\mathscr{A}_H$ which we will discuss this further in Lemma 3.12 below.

Moreover Lemma 3.12, together with Section 2.2, implies that for each critical point $(q,\tau)$ of $\mathscr{S}_L$, there exists a unique family $(q_s,\tau(s))\in\mathrm{Crit}(\mathscr{S}_{L+e(s)})$ for $s\in(-\varepsilon,\varepsilon)$, where $(q_0,\tau(0))=(q,\tau)$ and $e(0)=0$. Moreover we have $\tau'(0)\neq0$ and $e'(0)\neq0$. We can therefore define the **correction term**:

$$\chi(q,\tau):=\mathrm{sign}\left(-\frac{e'(0)}{\tau'(0)}\right)\in\{-1,1\}.$$

A proof of the following result can be found in [34, Section 10.2] (see also [35, Theorem 1.2]).

**3.8** Theorem. *Let $(q,\tau)\in\mathrm{Crit}(\mathscr{S}_L)$. Then*

$$m(q,\tau)=m_\tau(q)+\frac12-\frac12\chi(q,\tau).$$

## 3.6 The Palais-Smale condition

Work of Abbondandolo and Schwarz [5, 4] implies that we can find a smooth bounded vector field $\mathbf{G}$ on $\mathscr{P}_\alpha(M,S)\times\mathbb{R}^+$ with the following two properties:

- There exists a continuous function $\delta\in C^\infty(\mathscr{P}_\alpha(M,S)\times\mathbb{R}^+,\mathbb{R})$ such that for all $(q,\tau)\in\mathscr{P}_\alpha(M,S)\times\mathbb{R}^+$ one has

$$\mathrm{d}\mathscr{S}_L(q,\tau)(\mathbf{G}(q,\tau))\geq\delta(\mathscr{S}_L(q,\tau))\left\|\mathrm{d}\mathscr{S}_L(q,\tau)\right\|_{W^{1,2}}.$$

- For each $(q,\tau) \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$ one has
$$\mathscr{S}_L(q,\tau) \in \mathrm{Crit}(\mathscr{S}_L) \quad \Leftrightarrow \quad \mathbf{G}(q,\tau) = 0,$$
and moreover if $(q,\tau) \in \mathrm{Crit}(\mathscr{S}_L)$ then
$$\nabla^2 \mathscr{S}_L(q,\tau) = \nabla \mathbf{G}(q,\tau)$$
(where here $\nabla \mathbf{G}(q,\tau)$ denotes the **Jacobian** of $\mathbf{G}$, defined by $\nabla \mathbf{G}(q,\tau)(\xi,h) := [\mathbf{G},V](q,\tau)$, where $V$ is any vector field on $\mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$ such that $V(q,\tau) = (\xi,h)$).

Moreover in the case $\alpha = 0$, we may additionally insist that the following two properties hold:

- There exists $k_1 > 0$ such that
$$\left\langle\!\left\langle \mathbf{G}(q,\tau), \frac{\partial}{\partial \tau} \right\rangle\!\right\rangle_{W^{1,2}} < 0 \quad \text{if} \quad \mathscr{S}_L(q,\tau) \geq k_1 \tau \tag{3.5}$$
(see [4, Section 11] or [34, Lemma 10.3]).
- The submanifold $\underline{S} \times \mathbb{R}^+ \subseteq \mathscr{P}_0(M,S) \times \mathbb{R}^+$ is invariant under $G_s$, that is, whenever defined one has
$$G_s(\underline{S} \times \mathbb{R}^+) \subseteq \underline{S} \times \mathbb{R}^+ \subseteq \mathscr{P}_0(M,S) \times \mathbb{R}^+.$$

We shall refer to a smooth bounded vector field $\mathbf{G}$ that satisfies these four properties as a **refined pseudo-gradient** for $\mathscr{S}_L$. The next result is the key to defining the Morse (co)complex of $\mathscr{S}_L$ (compare [4, Proposition 11.1, Proposition 11.2]). A full proof in our setting is given in [34].

**3.9** Theorem. *Let $\mathbf{G}$ denote a refined pseudo-gradient for $\mathscr{S}_L$, and let $G_s$ denote the flow of $-\mathbf{G}$. Then:*

1. *If $\alpha \neq 0$ then the pair $(\mathscr{S}_L, \mathbf{G})$ satisfies the Palais-Smale condition at the level $a$ for all $a \in \mathbb{R}$. If $\alpha = 0$ then the pair $(\mathscr{S}_L, \mathbf{G})$ satisfies the Palais-Smale condition at the level $a$ for all $a \in \mathbb{R} \backslash \{0\}$.*
2. *$\mathscr{S}_L$ is bounded below on $\mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$.*
3. *If $\alpha \neq 0$ then $\omega_+(q,\tau) = \infty$ for all $(q,\tau) \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$. Moreover if $(q,\tau) \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$ and $(q_s,\tau_s) := G_s(q,\tau)$ then $\tau_s$ is bounded strictly away from zero as $s \to \infty$.*
4. *If $\alpha = 0$ and $(q,\tau) \in \mathscr{P}_0(M,S) \times \mathbb{R}^+$ satisfies $\omega_+(q,\tau) < \infty$, then if we define $(q_s,\tau_s) := G_s(q,\tau)$ one has $\mathscr{S}_L(q_s,\tau_s) \to 0$, $\tau_s \to 0$ and $q_s$ converges to a constant loop as $s \to \omega_+(q,\tau)$. If instead $\omega_+(q,\tau) = \infty$ then $\tau_s$ is strictly bounded away from zero as $s \to \infty$.*
5. *If $\alpha \neq 0$ then $\omega_-(q,\tau) = -\infty$ for all $(q,\tau) \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$.*
6. *Given $a > 0$ define*
$$\mathscr{O}(a) := \{(q,\tau) \in \mathscr{P}_0(M,S) \times (0,k_1 a) \,:\, \mathscr{S}_L(q,\tau) < a\},$$
*where $k_1 > 0$ was defined in (3.5). Then $\mathscr{O}(a) \cap \mathrm{Crit}(\mathscr{S}_L) = \emptyset$ for all $a > 0$, and for any $a > 0$, if $(q,\tau) \in \mathscr{O}(a)$ then $G_s(q,\tau) \in \mathscr{O}(a)$ for all $s \in (\omega_-(q,\tau),0]$. Finally if $(q,\tau) \in \mathscr{P}_0(M,S) \times \mathbb{R}^+$ is such that $\omega_-(q,\tau) > -\infty$ and $\mathscr{S}_L(q,\tau) \geq a$ then there exists $s < 0$ such that $G_s(q,\tau) \in \mathscr{O}(a)$.*

As in Section 1, where we abused the identification between a pair $(x,\tau)$ and the corresponding element $\zeta(t) := x(t/\tau)$ (see page 10), we shall start identifying a pair $(q,\tau) \in \mathscr{P}(M,S) \times \mathbb{R}^+$ with the path $\gamma : [0,\tau] \to M$ defined by $\gamma(t) := q(t/\tau)$. We will even do this for elements $(\underline{q},0) \in \underline{S} \times \{0\}$, where then $\gamma$ is the constant path $t \mapsto q$. As was with the case with Rabinowitz Floer homology, this should be viewed solely as a notational device (it is easier to write $\mathscr{W}(\gamma_-,\gamma_+;\ell)$ than

$\mathscr{W}((q_-,\tau_-),(q_+,\tau_+);\ell))$.

Given a refined pseudo-gradient $\mathbf{G}$ for $\mathscr{S}_L$ and a critical point $\gamma$ of $\mathscr{S}_L$, we denote by $\mathbf{W}^u(\gamma,-\mathbf{G})$ the **extended unstable manifold** of $(\gamma,-\mathbf{G})$. By definition $\mathbf{W}^u(\gamma,-\mathbf{G})$ is the union of the normal unstable manifold $W^u(\gamma,-\mathbf{G})$ together with the set of points one finds by following the forward orbit under $G_s$ of elements $\gamma'$ which do not converge in $\mathscr{P}(M,S)\times\mathbb{R}^+$ as $s\to\omega_+(\gamma')$. By Theorem 3.9.4 these are all of the form $(\underline{q},0)$ for some point $q\in S$. These are the so-called **critical points at infinity** in the sense of Bahri [15]).

In a similar vein it is convenient to define the following subset of $\mathscr{P}(M,S)\times[0,\infty)$:

$$\underline{\mathrm{Crit}}(\mathscr{S}_L):=\mathrm{Crit}(\mathscr{S}_L)\cup(\underline{S}\times\{0\})\,.$$

Our non-degeneracy assumption implies that the functional $\mathscr{S}_L$ is actually Morse, but it is not "Morse at infinity", in the sense that the critical points at infinity (i.e. the set $\underline{S}\times\{0\}$) form a Morse-Bott component of $\underline{\mathrm{Crit}}(\mathscr{S}_L)$. Thus we will need to work with flow lines with cascades in order to define the Morse (co)homology of $\mathscr{S}_L$, as we shall now explain.

## 3.7 The Morse complex

In order to define the Morse complex we will need three pieces of auxiliary data.

- Firstly, let $\mathbf{G}$ denote a refined pseudo-gradient for $\mathscr{S}_L$, and as before write $G_s$ for the local flow of $-\mathbf{G}$.
- Secondly, choose a Morse function $\ell:S\to\mathbb{R}$. In order to fit in with the approach taken in Section 2.4, it will be convenient to formally regard $\ell$ also as a function $\underline{\ell}:\underline{\mathrm{Crit}}(\mathscr{S}_L)\to\mathbb{R}$ by setting $\underline{\ell}(q,\tau):=0$ for $(q,\tau)\in\mathrm{Crit}(\mathscr{S}_L)$ and setting

  $$\underline{\ell}(\underline{q},0):=\ell(q)\quad\text{for }(\underline{q},0)\in\underline{S}\times\{0\}.$$

  We denote by $C(\ell)\subseteq\underline{\mathrm{Crit}}(\mathscr{S}_L)$ the set of critical points of $\underline{\ell}$ (so that $C(\ell)=\mathrm{Crit}(\mathscr{S}_L)\cup\mathrm{Crit}(\ell)$), and given $-\infty\le a<b\le\infty$ we define

  $$C(\ell)_a^b:=C(\ell)\cap\underline{\mathrm{Crit}}(\mathscr{S}_L)_a^b,$$

  where by definition $\mathscr{S}_L(\underline{q},0):=0$. It follows from Theorem 3.9 that if $b-a<\infty$ then $C(\ell)_a^b$ is always finite.
- Thirdly, let $g_\ell$ denote a Riemannian metric on $S$ such that the flow $\phi_t^\ell$ of $-\nabla\ell=-\nabla_{g_\ell}\ell$ is Morse-Smale. As with $\ell$, we can formally regard $\phi_t^\ell$ as a flow on $\underline{\mathrm{Crit}}(\mathscr{S}_L)$ by defining $\phi_t^\ell(q,\tau):=(q,\tau)$ for all $(q,\tau)\in\mathrm{Crit}(\mathscr{S}_L)$ and $t\in\mathbb{R}$.

Given $\gamma\in C(\ell)$, we denote by $i_\ell(\gamma)$ the Morse index of $\gamma$ seen as a critical point of $\underline{\ell}$, so that $i_\ell(\gamma):=\dim W^u(\gamma;-\nabla\ell)$. Thus $i_\ell(\gamma)=0$ unless $\gamma=(\underline{q},0)$ for some $q\in\mathrm{Crit}(\ell)$. Finally, let us define the integer valued function

$$m_\ell:C(\ell)\to\mathbb{Z}$$

by

$$m_\ell(\gamma):=m(\gamma)+i_\ell(\gamma),$$

where by definition $m(\underline{q},0):=0$.

The Morse complex is defined with the aid of the spaces $\mathscr{W}(\gamma_-,\gamma_+;\ell)$ of gradient flow lines with cascades running between two critical points $\gamma_-$ and $\gamma_+$ of $C(\ell)$. These spaces are be defined entirely analogously to the spaces $\mathscr{M}(\zeta_-,\zeta_+)$ from Definition 2.10, only we work with $\mathscr{S}_L$ and $\underline{\ell}$ rather than $\mathscr{A}_H$ and $h$. We use the letter $\mathscr{W}$ instead of $\mathscr{M}$ to help distinguish between the two, and we include the "$\ell$" in the notation because later on we will use these spaces with different choices of Morse function $\ell$.

The next theorem, together with Theorem 3.11 below, follows from Theorem 3.9 exactly as in [4, Section 11]. The main ingredients are Abbondandolo and Majer's infinite dimensional Morse theory [1] and Frauenfelder's cascades approach to Morse-Bott homology (as described in [26] and Section 2.4).

**3.10** THEOREM. *For a generic choice of* $\mathbf{G}$ *and* $g_\ell$ *the sets* $\mathscr{W}(\gamma_-, \gamma_+; \ell)$ *are all smooth manifolds of finite dimension*

$$\dim \mathscr{W}(\gamma_-, \gamma_+; \ell) = m_\ell(\gamma_-) - m_\ell(\gamma_+) - 1.$$

*Moreover if* $m_\ell(\gamma_-) - m_\ell(\gamma_+) = 1$ *then* $\mathscr{W}(\gamma_-, \gamma_+; \ell)$ *is compact, and hence a finite set.*

Denote by

$$\mathrm{CM}_*^\alpha(L, \ell)_a^b := C_*(\ell)_a^b \otimes \mathbb{Z}_2,$$

where the grading $*$ is given by the function $m_\ell$.

Given $\gamma_-, \gamma_+ \in C(\ell)_a^b$ with $m_\ell(\gamma_-) = m_\ell(\gamma_+) + 1$, we define the number $n_{\mathrm{Morse}}(\gamma_-, \gamma_+; \ell) \in \mathbb{Z}_2$ to be the parity of the finite set $\mathscr{W}(\gamma_-, \gamma_+; \ell)$. If $\gamma_+ \in C(\ell)_a^b$ has $m_\ell(\gamma_-) \neq m_\ell(\gamma_+) + 1$, set $n_{\mathrm{Morse}}(\gamma_-, \gamma_+) = 0$. Now we can define the boundary operator

$$\partial_a^b = \partial_a^b(L, \mathbf{G}, \ell, g_\ell) : \mathrm{CM}_*^\alpha(L, \ell)_a^b \to \mathrm{CM}_{*-1}^\alpha(L, \ell)_a^b$$

as the linear extension of

$$\partial_a^b \gamma_- := \sum_{\gamma_+ \in C(\ell)_a^b} n_{\mathrm{Morse}}(\gamma_-, \gamma_+; \ell)\gamma_+.$$

The next result is the **Morse homology theorem**. Let us write

$$\mathscr{S}_L^b(\alpha) := \left\{(q, \tau) \in \mathscr{P}_\alpha(M, S) \times \mathbb{R}^+ \ : \ \mathscr{S}_L(q, \tau) < b\right\}.$$

**3.11** THEOREM. *For a generic choice of* $\mathbf{G}$ *and* $g_\ell$*, it holds that* $\partial_a^b \circ \partial_a^b = 0$*. Thus* $\{\mathrm{CM}_*^\alpha(L, \ell), \partial_a^b\}$ *forms a chain complex. The isomorphism class of this complex is independent of the choice of* $\mathbf{G}$*,* $\ell$ *and* $g_\ell$*. The associated homology, known as the* **Morse homology** *of* $\mathscr{S}_L$*, is isomorphic to the singular (co)homology of the pair* $(\mathscr{S}_L^b(\alpha), \mathscr{S}_L^a(\alpha))$*.*

$$\mathrm{HM}_*^\alpha(L)_a^b \cong \mathrm{H}_*(\mathscr{S}_L^b(\alpha), \mathscr{S}_L^a(\alpha); \mathbb{Z}_2).$$

*In particular, if* $b = \infty$ *and* $a < \inf \mathscr{S}_L$ *then*

$$\mathrm{HM}_*^\alpha(L) := \mathrm{HM}_*^\alpha(L)_a^\infty \cong \mathrm{H}_*(\mathscr{P}_\alpha(M, S) \times \mathbb{R}^+; \mathbb{Z}_2) \cong \mathrm{H}_*(P_\alpha(M, S); \mathbb{Z}_2).$$

One can also play the same game with cohomology. For reasons that will become clear shortly, it is convenient to use the Morse function $-\ell$ when defining the Morse cohomology (but this is not necessary). Given $-\infty \leq a < b \leq \infty$, let $C(-\ell)_a^b$ denote the set of critical points $\gamma$ of $-\ell$ with $a < \mathscr{S}_L(\gamma) < b$. We grade $C(-\ell)$ with $m_{-\ell}$. Note that $C(\ell)_a^b = C(-\ell)_a^b$ as sets but in general not as graded sets: $C_m(\ell)_a^b \neq C_m(-\ell)_a^b$. Now set

$$\mathrm{CM}_\alpha^*(L, -\ell)_a^b := \prod_{\gamma \in C_*(-\ell)_a^b} \mathbb{Z}_2 \gamma,$$

and define

$$\delta_a^b = \delta_a^b(L, \mathbf{G}, -\ell, g_\ell) : \mathrm{CM}_\alpha^*(L, -\ell)_a^b \to \mathrm{CM}_\alpha^{*+1}(L, -\ell)_a^b$$

by

$$\delta_a^b \gamma_- := \sum_{\gamma_+ \in C_{*-1}(-\ell)_a^b} n_{\mathrm{Morse}}(\gamma_+, \gamma_-; -\ell)\gamma_+$$

(here $n_{\mathrm{Morse}}(\gamma_+, \gamma_-; -\ell)$ denotes the parity of the corresponding finite set $\mathscr{W}(\gamma_+, \gamma_-; -\ell)$).

Then $\delta_a^b \circ \delta_a^b = 0$, and hence $\{\mathrm{CM}_\alpha^*(L, -\ell)_a^b, \delta_a^b\}$ forms a cochain complex, whose cohomology computes the singular cohomology of the pair $(\mathscr{S}_L^b(\alpha), \mathscr{S}_L^a(\alpha))$.

### 3.8 Relating the two functionals $\mathscr{S}_L$ and $\mathscr{A}_H$

We will now study the relationship between the two functionals $\mathscr{S}_L$ and $\mathscr{A}_H$. The next lemma follows readily from the definitions.

**3.12** LEMMA. *The following relationships between* $\mathrm{Crit}(\mathscr{S}_L)$ *and* $\mathrm{Crit}(\mathscr{A}_H)$ *hold:*

1. *Given* $(q,\tau) \in \mathrm{Crit}(\mathscr{S}_L)$*, define*

$$\psi_+(q,\tau) := (x,\tau) \quad \text{where } x(t) := (q(t), \nabla L^{\mathsf{v}}(q,\dot{q}))\,.$$

$$\psi_-(q,\tau) := (\mathbb{I}(x), -\tau),$$

*where* $\mathbb{I}(x) := x(1-t)$*. Then if* $\alpha \neq 0$*, one has*

$$\mathrm{Crit}^{\alpha}(\mathscr{A}_H) = \psi_+\left(\mathrm{Crit}^{\alpha}(\mathscr{S}_L)\right) \cup \psi_-\left(\mathrm{Crit}^{-\alpha}(S_L)\right)$$

*and moreover one has*

$$\mathscr{A}_H(\psi_\pm(q,\tau)) = \pm \mathscr{S}_L(q,\tau).$$

2. *Given any* $(x,\tau) \in \mathscr{P}(X, N^*S) \times \mathbb{R}$ *with* $\tau > 0$*, if* $q := \pi \circ x$ *then*

$$\mathscr{A}_H(x,\tau) \leq \mathscr{S}_L(q,\tau),$$

$$\mathscr{A}_H(\mathbb{I}(x), -\tau) \geq -\mathscr{S}_L(q,\tau)$$

*with equality if and only if* $x = (q, \nabla L^{\mathsf{v}}(q,\dot{q}))$.

3. *Let* $(q,\tau) \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$ *denote a critical point of* $\mathscr{S}_L$*. Then for all* $(\xi,h) \in T_{\psi_+(q,\tau)}(\mathscr{P}_\alpha(X,N^*S) \times \mathbb{R})$ *it holds that*

$$\mathrm{d}^2\mathscr{A}_H(\psi_+(q,\tau))((\xi,h),(\xi,h)) \leq \mathrm{d}^2\mathscr{S}_L(q,\tau)((\xi^{\mathsf{h}},h),(\xi^{\mathsf{h}},h)),$$

*Let* $(q,\tau) \in \mathscr{P}_{-\alpha}(M,S) \times \mathbb{R}^+$ *denote a critical point of* $\mathscr{S}_L$*. Then for all* $(\xi,h) \in T_{\psi_-(q,\tau)}(\mathscr{P}_\alpha(X,N^*S) \times \mathbb{R})$ *it holds that*

$$\mathrm{d}^2\mathscr{A}_H(\psi_-(q,\tau))((\xi,h),(\xi,h)) \geq -\mathrm{d}^2\mathscr{S}_L(q,\tau)((\mathbb{I}(\xi)^{\mathsf{h}},-h),(\mathbb{I}(\xi)^{\mathsf{h}},-h)).$$

4. *Given a critical point* $(q,\tau)$*, a pair* $(\xi,h)$ *lies in the kernel of the Hessian of* $\mathscr{A}_H$ *at* $\psi_+(q,\tau)$ *[resp.* $\psi_-(q,\tau)$*] if and only if the pair* $(\xi^{\mathsf{h}},h)$ *[resp.* $(\mathbb{I}(\xi)^{\mathsf{h}},-h)$*] lies in the kernel of the Hessian of* $\mathscr{S}_L$ *at* $(q,\tau)$*.*

5. *If* $(q,\tau) \in \mathrm{Crit}(\mathscr{S}_L)$ *then*

$$\chi(q,\tau) = \chi(\psi_+(q,\tau)) = -\chi(\psi_-(q,\tau)).$$

Next, we discuss the relations between the indices of the critical points. We first recall the following statement, which is an extension of the **Morse index theorem** of Duistermaat [25] to the twisted symplectic form $\omega_\sigma$.

**3.13** THEOREM. **(The Morse index theorem)**
*Let* $(q,\tau) \in \mathrm{Crit}(\mathscr{S}_L)$*. Then*

$$m_\tau(q) = \mu_{\mathsf{Ma}}(\psi_-(q,\tau)) + d - \frac{n}{2}.$$

**Proof.** We deduce this from the equivalent statement for the standard symplectic form $d\lambda_0$ (specifically, from [2, Corollary 4.2]) by arguing as follows: take a tubular neighborhood $W$ of $q([0,1])$ in $M$. Since $H^2(W) = 0$, $\sigma|_W = d\theta$ for some $\theta \in \Omega^1(W)$. The flow $\phi_t^H|_W$ is conjugated to the flow $\psi_t^{H_\theta} : T^*W \to T^*W$, where $H_\theta(q,p) = H(q, p - \theta_q)$ and $\psi_t^{H_\theta}$ denotes the flow of the symplectic gradient of $H_\theta$ with respect to the standard symplectic form $d\lambda_0$. Since both the Maslov index and the Morse index are local invariants, the theorem now follows directly from [2, Corollary 4.2]. ■

Recall that in order to define the Lagrangian Rabinowitz Floer chain complex we need to pick a Morse function $h$ on $\mathrm{Crit}(\mathscr{A}_H)$. It is convenient to choose $h$ and $\underline{\ell}$ so that they satisfy the following properties.

1. For all $(q, \tau) \in \mathrm{Crit}(\mathscr{S}_L)$ one has
$$\underline{\ell}(q, \tau) = h(\psi_-(q, \tau)) = h(\psi_+(q, \tau)).$$
2. The function $\ell$ has a unique minimum $q_{\min}$ and a unique maximum $q_{\max}$ for two points $q_{\min}, q_{\max} \in S$ and is **self-indexing**, that is, $\ell(q) = i_\ell(q)$ for all $q \in \mathrm{Crit}(\ell)$. Note that if $d = \dim S = 0$ (i.e. $S = \{q\}$ and $N^*S = T_q^*M$) then we obviously have $q_{\min} = q_{\max} = q$, but that in all other cases clearly $q_{\min} \neq q_{\max}$.
3. For all $x \in \Sigma \cap N^*S$, we have $\ell(\pi(x)) \leq h(\underline{x}, 0) \leq \ell(\pi(x)) + 1/2$.
4. Every critical point of $h|_{(\underline{\Sigma \cap N^*S}) \times \{0\}}$ lies above a critical point of $\ell$ and moreover for each critical point $q$ of $\ell$ there are exactly two critical points of $h|_{(\underline{\Sigma \cap N^*S}) \times \{0\}}$ in $\underline{\Sigma \cap T_q^*M} \times \{0\}$. Denoting these two critical points by $\psi_\pm(\underline{q}, 0)$, it holds that
$$\ell(q) = h(\psi_-(\underline{q}, 0)) = h(\psi_+(\underline{q}, 0)) - 1/2,$$
$$i_\ell(q) = i_h(\psi_-(\underline{q}, 0)) = i_h(\psi_+(\underline{q}, 0)) - n + d + 1.$$

That such functions exist is explained in detail in [4, Appendix B]. With this choice of functions $h$ and $\ell$ the following relationships hold - the proof is an immediate application of Theorem 3.8, Lemma 3.12.5, and Theorem 3.13.

**3.14** COROLLARY. **(The relationship between the indices of $C(\ell)$ and $C(h)$)**
*Let $\gamma = (q, \tau) \in C(\ell)$ and let $\zeta_+ := \psi_+(\gamma)$ and $\zeta_- := \psi_-(\gamma)$.*
*Then*
$$m_\ell(\gamma) = \begin{cases} \mu_h(\zeta_+), & \tau > 0, \\ -\mu_h(\zeta_-) + 2d - n + 1, & \tau > 0, \\ \mu_h(\zeta_+), & \tau = 0, \\ \mu_h(\zeta_-) - d + n - 1, & \tau = 0 \end{cases}$$
*and*
$$m_{-\ell}(\gamma) = \begin{cases} \mu_h(\zeta_+), & \tau > 0, \\ -\mu_h(\zeta_-) + 2d - n + 1, & \tau > 0, \\ -\mu_h(\zeta_+) + d, & \tau = 0, \\ -\mu_h(\zeta_-) + 2d - n + 1, & \tau = 0. \end{cases}$$

## 3.9 The Main Theorem

In this section we state the main result of this thesis, which is the extension of [4, Theorem 2] to our setting.

**3.15** THEOREM. **(Computation of the Lagrangian Rabinowitz Floer homology)**
*Let $h : \mathrm{Crit}(\mathscr{A}_H) \to \mathbb{R}$ and $\ell : S \to \mathbb{R}$ be Morse functions as specified above. Let $g_h$ and $g_\ell$ denote generically chosen Riemannian metrics on $\mathrm{Crit}(\mathscr{A}_H)$ and $S$ respectively, such that the flows $\phi_t^h$ and $\phi_t^\ell$ of $-\nabla h = -\nabla_{g_h} h$ and $-\nabla \ell = -\nabla_{g_\ell} \ell$ are Morse-Smale. Let $\mathbf{G}$ denote a generically chosen refined pseudo-gradient for $\mathscr{S}_L$, and let $\mathbf{J} = (J_t) \subseteq \mathscr{J}(X, \omega_\sigma)$ denote a generic family of almost complex structures, such that $\sup_t \|J_t - J_g\|_{L^\infty}$ is sufficiently small (this is needed to obtain the required $L^\infty$-estimates, cf. Section 3.2).*

*Fix $-\infty < a < b < \infty$. Then there exists:*

1. *An injective chain map*

$$(\Phi_{\mathsf{SA}})_a^b : \mathrm{CM}_*^\alpha(L,\ell)_a^b \to \mathrm{CRF}_*^\alpha(H,h)_a^b$$

*which admits a left inverse* $(\widehat{\Phi}_{\mathsf{SA}})_a^b : \mathrm{CRF}_*^\alpha(H,h)_a^b \to \mathrm{CM}_*^\alpha(L,\ell)_a^b$.

2. *A surjective chain map*

$$(\Phi_{\mathsf{AS}})_a^b : \mathrm{CRF}_*^\alpha(H,h)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a}$$

*which admits a right inverse* $(\widehat{\Phi}_{\mathsf{AS}})_a^b : \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a} \to \mathrm{CRF}_*^\alpha(H,h)_b^a$.

*Moreover:*

1. *If $d < n/2$ then for degree reasons alone we deduce that $\Phi_{\mathsf{SA}}$ and $\Phi_{\mathsf{AS}}$ define chain complex isomorphisms*

$$(\Phi_{\mathsf{SA}})_a^b : \mathrm{CM}_*^\alpha(L,\ell)_a^b \cong \mathrm{CRF}_*^\alpha(H,h)_a^b,$$

$$(\Phi_{\mathsf{AS}})_a^b : CRF_*^\alpha(H,h)_a^b \to CM_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a}.$$

*Thus in the limit $a \to -\infty$, $b \to \infty$, if we identify $\mathrm{HM}_*^\alpha(L,\ell) \cong \mathrm{H}_*(P_\alpha(M,S);\mathbb{Z}_2)$ and $\mathrm{HM}_{-\alpha}^*(L,-\ell) \cong \mathrm{H}^*(P_{-\alpha}(M,S);\mathbb{Z}_2)$ we deduce that*

$$\mathrm{RFH}_*^\alpha(\Sigma, N^*S, X) \cong \begin{cases} \mathrm{H}_*(P_\alpha(M,S);\mathbb{Z}_2), & * \geq 0, \\ 0, & 2d-n+1 < * < 0, \\ \mathrm{H}^{-*+2d-n+1}(P_{-\alpha}(M,S);\mathbb{Z}_2), & * \leq 2d-n+1. \end{cases}$$

2. *If $\alpha \neq 0$ and $d \geq n/2$, or if $\alpha = 0$ and $d = n/2$ with $n \geq 4$ then the composition $(\Phi_{\mathsf{AS}})_a^b \circ (\Phi_{\mathsf{SA}})_a^b : \mathrm{CM}_*^\alpha(L,\ell)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a}$ is chain homotopic to zero, that is, there exists a homomorphism*

$$\Theta_a^b : \mathrm{CM}_*^\alpha(L,\ell)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n}(L,-\ell)_{-b}^{-a}$$

*such that*

$$(\Phi_{\mathsf{AS}})_a^b \circ (\Phi_{\mathsf{SA}})_a^b = \Theta_a^b \circ \partial_a^b + \delta_{-b}^{-a} \circ \Theta_a^b.$$

*Setting*

$$\Psi_a^b := (\Phi_{\mathsf{SA}})_a^b - (\widehat{\Phi}_{\mathsf{AS}})_a^b \circ \Theta_a^b \circ \partial_a^b - \partial_a^b \circ (\widehat{\Phi}_{\mathsf{AS}})_a^b \circ \Theta_a^b,$$

*the map $\Psi_a^b$ is chain homotopic to $(\Phi_{\mathsf{SA}})_a^b$, and satisfies $(\Phi_{\mathsf{AS}})_a^b \circ \Psi_a^b = 0$. Thus we obtain a short exact sequence of chain complexes*

$$0 \to \mathrm{CM}_*^\alpha(L,\ell)_a^b \xrightarrow{\Psi_a^b} \mathrm{CRF}_*^\alpha(H,h)_a^b \xrightarrow{(\Phi_{\mathsf{AS}})_a^b} \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a} \to 0.$$

*Thus in the limit $a \to -\infty$, $b \to \infty$, if we identify $\mathrm{HM}_*^\alpha(L,\ell) \cong \mathrm{H}_*(P_\alpha(M,S);\mathbb{Z}_2)$ and $\mathrm{HM}_{-\alpha}^*(L,-\ell) \cong \mathrm{H}^*(P_{-\alpha}(M,S);\mathbb{Z}_2)$, then we obtain the long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}_i(P_\alpha(M,S);\mathbb{Z}_2) \arrow[r, "\Theta_*"] & \mathrm{RFH}_i^\alpha(\Sigma, N^*S, X) \arrow[d, "(\Psi_{\mathrm{AS}})_*"] & & \\ & & \mathrm{H}^{-i+2d-n+1}(P_{-\alpha}(M,S);\mathbb{Z}_2) \arrow[r, "\Delta"] & \mathrm{H}_{i-1}(P_\alpha(M,S);\mathbb{Z}_2) \arrow[r] & \cdots \end{tikzcd}$$

*The connecting homomorphism $\Delta$ is identically zero unless $\alpha = 0$ and $i = 1$, in which case it is given by (recall by assumption when $\alpha = 0$ one has $d = n/2$):*

$$\begin{tikzcd} \mathrm{H}^0(P_0(M,S);\mathbb{Z}_2) \arrow[r, "\Delta"] \arrow[d] & \mathrm{H}_0(P_0(M,S);\mathbb{Z}_2) \\ \mathrm{H}^0(S;\mathbb{Z}_2) \arrow[r, "f"'] & \mathrm{H}_0(S;\mathbb{Z}_2) \arrow[u] \end{tikzcd}$$

*where*
$$f(c) := \mathrm{PD}(c \smile e(N^*S)),$$
*($e(N^*S)$ is the Euler class of $N^*S \to S$) and the vertical maps are the isomorphisms induced by the inclusion $S \hookrightarrow \underline{S} \subseteq P_0(M,S)$.*

The proof of this theorem is very similar to the corresponding proof in [4]. We will therefore omit large swathes of the technical details, referring the reader to the beautiful and lucid exposition in [4], and instead just give an outline of Abbondandolo and Schwarz' constructions. More details specific to the Lagrangian case we study here can be found in [34].

## 3.10 The extended unstable manifolds with cascades $\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell)$

Recall the definition of the extended unstable manifold $\mathbf{W}^u(\gamma; -\mathbf{G})$ introduced on page 27. We now introduce the **extended unstable manifold with cascades**, which we denote by $\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell)$.

Fix $\gamma \in C(\ell)$. If $m \in \mathbb{N}$, let $\widetilde{\mathscr{W}}^u_m(\gamma)$ denote the set of tuples $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_m)$ such that $\gamma_i \in \mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$ for $i = 1, \dots, m-1$ and $\gamma_m \in (\mathscr{P}_\alpha(M,S) \times \mathbb{R}^+) \cup (\underline{S} \times \{0\})$ with

$$\gamma_1 \in \mathbf{W}^u(W^u(\gamma; -\nabla\ell); -\mathbf{G}),$$

$$\lim_{s \to -\infty} G_s(\gamma_{i+1}) \in \phi^{-\nabla\ell}_{[0,\infty)}\left(\lim_{s\to\infty} G_s(\gamma_i)\right) \quad \text{for } i = 1, \dots, m-1.$$

Of course, if $\alpha \neq 0$ then $\gamma_m$ is always in $\mathscr{P}_\alpha(M,S) \times \mathbb{R}^+$.

Let $\mathscr{W}^u_m(\gamma)$ denote the quotient of $\widetilde{\mathscr{W}}^u_m(\gamma)$ under the free $\mathbb{R}^{m-1}$ action given by

$$(\gamma_1, \dots, \gamma_{m-1}) \mapsto (G_{s_1}(\gamma_1), \dots, G_{s_{m-1}}(\gamma_{m-1})), \quad (s_1, \dots, s_{m-1}) \in \mathbb{R}^{m-1}.$$

Then put
$$\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell) := \bigcup_{m \in \mathbb{N}\cup\{0\}} \mathscr{W}^u_m(\gamma).$$

Note there is a well defined evaluation map

$$\mathrm{ev} : \mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell) \to \left(\mathscr{P}_\alpha(M,S) \times \mathbb{R}^+\right) \cup (\underline{S} \times \{0\})],$$

given by
$$\mathrm{ev}(\boldsymbol{\gamma}) := \gamma_m \quad \text{for} \quad \boldsymbol{\gamma} \in \mathscr{W}^u_m(\gamma).$$

For a generic choice of $\mathbf{G}$ and $g_\ell$ the spaces $\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell)$ admit the structure of smooth manifolds of finite dimension

$$\dim \mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell) = m_\ell(\gamma).$$

This can be proved using [26, Corollary A.16], and details can be found in [34, Section 12.1].

## 3.11 The chain map $\Phi_{\mathsf{SA}}$

In this section we define a chain map

$$(\Phi_{\mathsf{SA}})^b_a : \mathrm{CM}^\alpha_*(L,\ell)^b_a \to \mathrm{CRF}^\alpha_*(H,h)^b_a.$$

In order to define the chain map $\Phi_{\mathrm{SA}}$, one first needs to construct a suitable moduli space. Here are the details.

We will need to study the space of **positive half flow lines with cascades** for $\mathscr{A}_H$, denoted by $\mathbf{M}^s(\zeta)$. Fix $\zeta \in C(h)$. Given $m \in \mathbb{N}$ let $\widetilde{\mathscr{M}}^s_m(\zeta)$ denote the denote the set of $m$-tuples of maps $\boldsymbol{u} = (u_1, \dots, u_m)$ such that
$$u_1 : [0,\infty) \to P_\alpha(X, N^*S) \times \mathbb{R};$$

$$u_2, \dots, u_m : \mathbb{R} \to P_\alpha(X, N^*S) \times \mathbb{R},$$

are all gradient flow lines of $(H, \mathbf{J})$ (which are possibly stationary solutions) and such that

$$\lim_{s\to\infty} u_m(s) \in W^s(\zeta; -\nabla h);$$

$$\lim_{s\to-\infty} u_{i+1}(s) \in \phi^h_{[0,\infty)}(\lim_{s\to\infty} u_i(s)) \quad \text{for } i = 1, \dots, m-1.$$

Let $\mathscr{M}^s_m(\zeta)$ denote the quotient of $\widetilde{\mathscr{M}^s_m}(\zeta)$ under the free $\mathbb{R}^{m-1}$ action given by translation along the flow lines $u_2, \dots, u_m$.

Then put

$$\mathbf{M}^s(\zeta) := \bigcup_{m\in\mathbb{N}} \mathscr{M}^s_m(\zeta).$$

The space $\mathbf{M}^s(\zeta)$ is not finite dimensional. However, by restricting where $u_1$ can "begin", we can cut it down to something finite dimensional. This is precisely what the moduli space $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ does. Fix $\gamma \in C(\ell)$ and $\zeta \in C(h)$. The moduli space $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ is defined to be the following subset of $\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell) \times \mathbf{M}^s(\zeta)$. Namely, an element

$$(\boldsymbol{\gamma}, \mathbf{u}) \in \mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell) \times \mathbf{M}^s(\zeta)$$

with $\boldsymbol{\gamma} \in \mathscr{W}^u_m(\gamma)$ lies in $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ if and only if, writing $u_1 = (x, \tau)$ one has

$$\gamma_m = (\pi \circ x_1(0, \cdot), \tau_1(0)).$$

In other words, $u_1$ must "begin" over $\gamma_m$.

This defines a Lagrangian boundary condition. This implies that we have a Fredholm problem, and since generically $\mathbf{W}^u(\gamma; -\mathbf{G}, -\nabla\ell)$ is a finite dimensional manifold, it follows that $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ can be seen as the zero set of a Fredholm operator. In fact, more is true.

**3.16** THEOREM. *For a generic choice of* $\mathbf{G}$*,* $\mathbf{J}$*,* $g_\ell$ *and* $g_h$*, the spaces* $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ *are precompact smooth manifolds of finite dimension*

$$\dim \mathscr{M}_{\mathsf{SA}}(\gamma, \zeta) = m_\ell(\gamma) - \mu_h(\zeta).$$

**Proof.** The only complication with obtaining transversality is the presence of stationary solutions, which can appear if $\zeta = \psi_+(\gamma)$ or $\gamma = (\underline{q}, 0)$ and $\zeta = \psi_\pm(\underline{q}, 0)$. In the former case the first inequality of the third statement of Lemma 3.12 forces the linearized operator defining the moduli space $\mathscr{M}_{\mathsf{SA}}(\gamma, \psi_+(\gamma))$ to be an isomorphism (see [4, Lemma 6.2] or [3, Proposition 3.7]), and in the second two cases the four assumptions made earlier on the Morse functions $h$ and $\underline{\ell}$ guarantee that the linearized operator defining the moduli spaces $\mathscr{M}_{\mathsf{SA}}(\gamma, \psi_\pm(\gamma))$ is surjective (see [4, Lemma 6.3]).

The index computation can be proved by combining [6, Theorem 5.24] (a special case of this is given in [2, Proposition 7.3]) and the arguments from [18, Section 4]. Full details can be found in [34, Theorem 12.3].

Finally we address the precompactness statement. The key point here is the following chain of inequalities, which follows from Lemma 3.12.2:

$$\begin{aligned} \mathscr{S}_L(\gamma) \geq \mathscr{S}_L(\gamma_i) \geq \mathscr{S}_L(\gamma_m) = \mathscr{S}_L(\pi \circ x_1(0, \cdot), \tau_1(0)) \\ \geq \mathscr{A}_H(u_1(0, \cdot)) \geq \mathscr{A}_H(u_i(s, \cdot)) \geq \mathscr{A}_H(\zeta). \end{aligned} \tag{3.6}$$

More details can be found in [4, Section 6] (see also [34, Theorem 12.3]). ■

Putting this together, we deduce that when $m_\ell(\gamma) = \mu_h(\zeta)$, the moduli space $\mathscr{M}_{\mathsf{SA}}(\gamma, \zeta)$ is a finite set, and hence we can define $n_{\mathsf{SA}}(\gamma, \zeta) \in \mathbb{Z}_2$ to be its parity. If $m_\ell(\gamma) \neq \mu_h(\zeta)$ then set $n_{\mathsf{SA}}(\gamma, \zeta) = 0$.

Then one defines $(\Phi_{\mathsf{SA}})_a^b : \mathrm{CM}_*^\alpha(L,\ell)_a^b \to \mathrm{CRF}_*^\alpha(H,h)_a^b$ as the linear extension of

$$(\Phi_{\mathsf{SA}})_a^b\gamma = \sum_{\zeta \in C(h)_a^b} n_{\mathsf{SA}}(\gamma,\zeta)\zeta$$

(we are implicitly using (3.6) here to ensure that the choice of action window makes sense). A standard gluing argument shows that $(\Phi_{\mathsf{SA}})_a^b$ is a chain map.

## 3.12 The chain map $\Phi_{\mathsf{AS}}$

In this section we define a chain map

$$(\Phi_{\mathsf{AS}})_a^b : \mathrm{CRF}_*^\alpha(H,h)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L,-\ell)_{-b}^{-a}].$$

It is defined in much the same way. One begins by defining a space $\mathbf{M}^u(\zeta)$ of **negative half flow lines with cascades**. Given $m \in \mathbb{N}$ let $\widetilde{\mathscr{M}}_m^u(\zeta)$ denote the denote the set of tuples of maps $\boldsymbol{u} = (u_1,\dots,u_m)$ such that

$$u_1,\dots,u_{m-1} : \mathbb{R} \to P_\alpha(X,N^*S)\times\mathbb{R};$$

$$u_m : (-\infty,0] \to P_\alpha(X,N^*S)\times\mathbb{R},$$

which are gradient flow lines of $(H,\mathbf{J})$ (which are possibly stationary solutions) and such that

$$\lim_{s\to-\infty} u_m(s) \in W^u(\zeta;-\nabla h),$$

and such that

$$\lim_{s\to-\infty} u_{i+1}(s) \in \phi^h_{[0,\infty)}(\lim_{s\to\infty} u_i(s)) \quad \text{for } i=1,\dots,m-1.$$

Let $\mathscr{M}_m^u(\zeta)$ denote the quotient of $\widetilde{\mathscr{M}}_m^u(\zeta)$ under the free $\mathbb{R}^{m-1}$ action and put

$$\mathbf{M}^u(\zeta) := \bigcup_{m\in\mathbb{N}} \mathscr{M}_m^u(\zeta).$$

Now if $\zeta \in C(h)$ and $\gamma \in C(-\ell)$, we define $\mathscr{M}_{\mathsf{AS}}(\zeta,\gamma)$ to be the following subset of $\mathbf{W}^u(\gamma;-\mathbf{G},\nabla\ell)\times\mathbf{M}^u(\zeta)$ (note here we are using the Morse function $-\ell$). Namely, an element

$$(\boldsymbol{\gamma},\mathbf{u}) \in \mathbf{W}^u(\gamma;-\mathbf{G},\nabla\ell)\times\mathbf{M}^u(\zeta)$$

with $\boldsymbol{\gamma} \in \mathscr{W}_m^u(\zeta)$ and $\mathbf{u} \in \mathscr{M}_n^u(\zeta)$ lies in $\mathscr{M}_{\mathsf{AS}}(\zeta,\gamma)$ if and only if, writing $u_n = (x_n,\tau_n)$ one has

$$\gamma_m = (\pi\circ x_n(0,-\cdot),-\tau_n(0)).$$

The following theorem is proved in the same way as Theorem 3.16. Details can be found in [4, Section 9] and [34, Section 12.3].

**3.17** Theorem. *For a generic choice of* $\mathbf{G}$, $\mathbf{J}$, $g_\ell$ *and* $g_h$, *the spaces* $\mathscr{M}_{\mathsf{AS}}(\zeta,\gamma)$ *are precompact smooth manifolds of finite dimension*

$$\dim \mathscr{M}_{\mathsf{AS}}(\zeta,\gamma) = \mu_h(\zeta) + m_{-\ell}(\gamma) + n - 2d - 1.$$

We remark only that this time the key inequality responsible for compactness is the following: if $(\boldsymbol{\gamma},\mathbf{u}) \in \mathscr{M}_{\mathsf{AS}}(\zeta,\gamma)$ with $\boldsymbol{\gamma} \in \mathscr{W}_m^u(\zeta)$ and $\mathbf{u} \in \mathscr{M}_n^u(\zeta)$ then

$$\begin{aligned}\mathscr{A}_H(\zeta) &\geq \mathscr{A}_H(u_i(s,\cdot)) \geq \mathscr{A}_H(u_n(0,\cdot))\\ &\geq -\mathscr{S}_L(\pi\circ x_m(0,-\cdot),-\tau_m(0)) \geq -\mathscr{S}_L(\gamma_m) \geq -\mathscr{S}_L(\gamma_i) \geq -\mathscr{S}_L(\gamma).\end{aligned} \tag{3.7}$$

Putting this together, we deduce that when $\mu_h(\zeta) = -m_{-\ell}(\gamma) + 2d - n + 1$, the moduli space $\mathscr{M}_{\mathsf{AS}}(\zeta, \gamma)$ is a finite set, and hence we can define $n_{\mathsf{AS}}(\zeta, \gamma) \in \mathbb{Z}_2$ to be its parity. If $\mu_h(\zeta) \neq -m_{-\ell}(\gamma) + 1 - n + 2d$ then set $n_{\mathsf{AS}}(\zeta, \gamma) = 0$.

Then one defines $(\Phi_{\mathsf{AS}})_a^b : \mathrm{CRF}_*^\alpha(H, h)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n+1}(L, -\ell)_{-b}^{-a}$ as the linear extension of

$$(\Phi_{\mathsf{AS}})_a^b \zeta = \sum_{\gamma \in C(\ell)_{-b}^{-a}} n_{\mathsf{AS}}(\zeta, \gamma)\gamma,$$

(this time we are implicitly using (3.7) here to ensure that the choice of action window makes sense). A standard gluing argument shows that $(\Phi_{\mathsf{AS}})_a^b$ is a chain map.

## 3.13 The chain homotopy $\Theta$

**We assume throughout this section that $d \geq n/2$, and if $\alpha = 0$ then we additionally assume $n \geq 4$ and that $d = n/2$.**

We will construct a chain homotopy

$$\Theta_a^b : \mathrm{CM}_*^\alpha(L, \ell)_a^b \to \mathrm{CM}_{-\alpha}^{-*+2d-n}(L, -\ell)_{-b}^{-a}$$

which will have the property that

$$(\Phi_{\mathsf{AS}})_a^b \circ (\Phi_{\mathsf{SA}})_a^b = \Theta_a^b \circ \partial_a^b + \delta_{-b}^{-a} \circ \Theta_b^a.$$

This will involve counting a slightly different sort of object.

Let $\mathscr{F}_0$ denote the set of pairs $(u, R)$ where $R \in \mathbb{R}^+$ and $u = (x, \tau) : [-R, R] \times [0, 1] \to X \times \mathbb{R}$ satisfies the Rabinowitz Floer equation. Given $m \geq 1$, let $\widetilde{\mathscr{F}}_m$ denote the set of tuples $(v, \mathbf{u}, w)$ where $\mathbf{u} = (u_1, \dots, u_{m-1})$ are gradient flow lines of $\mathscr{A}_H$ such that

$$\lim_{s \to -\infty} u_{i+1}(s) \in \phi_{[0,\infty)}^h \left( \lim_{s \to \infty} u_i(s) \right) \quad \text{for } i = 1, \dots, m-2.$$

Next,

$$v : [0, \infty) \to P_\alpha(X, N^*S) \times \mathbb{R},$$
$$w : (-\infty, 0] \to P_\alpha(X, N^*S) \times \mathbb{R}$$

both satisfy the Rabinowitz Floer equation, with

$$\lim_{s \to -\infty} u_1(s) \in \phi_{[0,\infty)}^h \left( \lim_{s \to \infty} v(s) \right), \quad \lim_{s \to -\infty} w(s) \in \phi_{[0,\infty)}^h \left( \lim_{s \to \infty} u_{m-1}(s) \right).$$

Let $\mathscr{F}_m$ denote the quotient of $\widetilde{\mathscr{F}}_m$ by dividing through by the $\mathbb{R}^{m-1}$ action on the curves $u_1, \dots, u_{m-1}$.

Put

$$\mathscr{F} = \bigcup_{m \in \mathbb{N} \cup \{0\}} \mathscr{F}_m.$$

Given $\gamma_- \in C^\alpha(\ell)$ and $\gamma_+ \in C^{-\alpha}(\ell)$, we define $\mathscr{M}_\Theta(\gamma_-, \gamma_+)$ as the subset of

$$\mathbf{W}^u(\gamma_-; -\mathbf{G}, -\nabla \ell) \times \mathscr{F} \times \mathbf{W}^u(\gamma_+; -\mathbf{G}, \nabla \ell)$$

satisfying:

1. If $(\boldsymbol{\gamma}, (u, R), \boldsymbol{\gamma}') \in \mathscr{M}_\Theta(\gamma_-, \gamma_+)$ with $(u, R) \in \mathscr{F}_0$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_i) \in \mathbf{W}^u(\gamma_-; -\mathbf{G}, -\nabla \ell)$ and $\boldsymbol{\gamma}' = (\gamma_1', \dots, \gamma_j') \in \mathbf{W}^u(\gamma_+; -\mathbf{G}, \nabla \ell)$ then writing $u = (x, \tau)$, we require that

$$(\pi \circ x(-R, \cdot), \tau(-R)) = \gamma_i, \quad (\pi \circ x(R, -\cdot), -\tau(R)) = \gamma_j'.$$

2. If $(\boldsymbol{\gamma},(v,\mathbf{u},w),\boldsymbol{\gamma}') \in \mathscr{M}_\Theta(\gamma_-,\gamma_+)$ with $(v,\mathbf{u},w) \in \mathscr{F}_m$ for some $m \geq 1$, and $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_i) \in \mathbf{W}^u(\gamma_-;-\mathbf{G},-\nabla\ell)$ and $\boldsymbol{\gamma}' = (\gamma_1',\dots,\gamma_j') \in \mathbf{W}^u(\gamma_+;-\mathbf{G},\nabla\ell)$, then writing $v=(x,\tau)$ and $w=(x',\tau')$, we require that

$$(\pi \circ x(0,\cdot),\tau(0)) = \gamma_i, \quad (\pi \circ x'(0,-\cdot),-\tau'(0)) = \gamma_j'.$$

Let us note if $(\boldsymbol{\gamma},(u,R),\boldsymbol{\gamma}') \in \mathscr{M}_\Theta(\gamma_-,\gamma_+)$ with $(u,R) \in \mathscr{F}_0$, then if we write $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_i)$ and $\boldsymbol{\gamma}' = (\gamma_1',\dots,\gamma_j')$ then

$$\mathscr{S}_L(\gamma_-) \geq \mathscr{S}_L(\gamma_i) \geq \mathscr{A}_H(u(-R,\cdot)) \geq \mathscr{A}_H(u(s,\cdot)) \geq \mathscr{A}_H(u(R,\cdot)) \geq -\mathscr{S}_L(\gamma_j') \geq -\mathscr{S}_L(\gamma_+). \quad (3.8)$$

Similarly if $(\boldsymbol{\gamma},(v,\mathbf{u},w),\boldsymbol{\gamma}') \in \mathscr{M}_\Theta(\gamma_-,\gamma_+)$ with $\mathbf{u} \in \mathscr{F}_m$ for some $m \geq 1$, then if we write $\mathbf{u} = (u_1,\dots,u_m)$, $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_i)$ and $\boldsymbol{\gamma}' = (\gamma_1',\dots,\gamma_j')$ then

$$\mathscr{S}_L(\gamma_-) \geq \mathscr{S}_L(\gamma_i) \geq \mathscr{A}_H(v(0,\cdot)) \geq \mathscr{A}_H(u_i(s,\cdot)) \geq \mathscr{A}_H(w(0,\cdot)) \geq -\mathscr{S}_L(\gamma_j') \geq -\mathscr{S}_L(\gamma_+). \quad (3.9)$$

This time we have the following result. For more details we refer the reader to [4, Section 8] or [34, Section 12.4]. The latter reference explains exactly where the assumption that $d = n/2$ with $n \geq 4$ if $\alpha = 0$ is used.

**3.18** Theorem. *Denote by $C^\alpha_\Theta(\ell,-\ell) \subseteq C^\alpha(\ell) \times C^{-\alpha}(\ell)$ the set of pairs $(\gamma_-,\gamma_+)$ of critical points that satisfy*

$$m_\ell(\gamma_-) + m_{-\ell}(\gamma_+) \in \{2d-n, 2d-n+1\}.$$

*Then for a generic choice of $\mathbf{G}$, $\mathbf{J}$, $g_\ell$ and $g_h$, the spaces $\mathscr{M}_\Theta(\gamma_-,\gamma_+)$ for $(\gamma_-,\gamma_+) \in C^\alpha_\Theta(\ell,-\ell)$ are precompact smooth manifolds of finite dimension*

$$\dim \mathscr{M}_\Theta(\gamma_-,\gamma_+) = m_\ell(\gamma_-) + m_{-\ell}(\gamma_+) + n - 2d.$$

Now we move onto the key proposition which implies Theorem 3.15. The first statement of Theorem 3.19 below shows that under our assumptions, if we are given $\gamma_- \in C^\alpha(\ell)$ and $\gamma_+ \in C^{-\alpha}(\ell)$ with $m_\ell(\gamma_-) + m_{-\ell}(\gamma_+) = 2d-n$ then we can define $n_\Theta(\gamma_-,\gamma_+)$ as the parity of the finite set $\mathscr{M}_\Theta(\gamma_-,\gamma_+)$. This defines the chain map $\Theta_a^b$ (this time we are implicitly using (3.9) and (3.8) in order to ensure that the choice of action window makes sense). The fact that $\Theta_a^b$ is a chain homotopy between $(\Phi_{\mathsf{SA}})_a^b$ and $(\Phi_{\mathsf{AS}})_a^b$ involves studying the compactification of $\mathscr{M}_\Theta(\gamma_-,\gamma_+)$ by adding in the broken trajectories, and is the content of the second statement of the proposition below, which is taken from [4, Proposition 8.1]. Details of the proof in the Lagrangian case we study here can be found in [34, Section 12.10].

**3.19** Proposition. *Fix critical points $\gamma_- \in C_i^\alpha(\ell)_a^b$ and $\gamma_+ \in C_j^{-\alpha}(-\ell)_{-b}^{-a}$. Recall we always assume $d \geq n/2$ in this section, and if $\alpha = 0$ then we require $d = n/2$ and $n \geq 4$.*

1. *If $i+j = 2d-n$ then the moduli space $\mathscr{M}_\Theta(\gamma_-,\gamma_+)$ is compact.*
2. *If $i+j = 2d-n+1$ then the moduli space $\mathscr{M}_\Theta(\gamma_-,\gamma_+)$ is precompact, and we can identify the boundary $\partial\overline{\mathscr{M}}_\Theta(\gamma_-,\gamma_+)$ of the compactification $\overline{\mathscr{M}}_\Theta(\gamma_-,\gamma_+)$ as follows:*

$$\begin{aligned}\partial\overline{\mathscr{M}}_\Theta(\gamma_-,\gamma_+) \;=\; & \left\{ \bigcup_{\zeta \in C_i^\alpha(h)_a^b} \mathscr{M}_{\mathsf{SA}}(\gamma_-,\zeta) \times \mathscr{M}_{\mathsf{AS}}(\zeta,\gamma_+) \right\} \\ & \bigcup \left\{ \bigcup_{\gamma \in C_{i-1}^\alpha(\ell)_b^a} \mathscr{W}(\gamma_-,\gamma;\ell) \times \mathscr{M}_\Theta(\gamma,\gamma_+) \right\} \\ & \bigcup \left\{ \bigcup_{\gamma' \in C_{j-1}^\alpha(-\ell)_{-b}^{-a}} \mathscr{M}_\Theta(\gamma_-,\gamma') \times \mathscr{W}(\gamma_+,\gamma';-\ell) \right\}.\end{aligned}$$

Theorem 3.15 essentially follows from this proposition; see [4, Section 9] for the details.

## References


[1] A. Abbondandolo and P. Majer. Lectures on the Morse complex for infinite dimensional manifolds. In P Biran, O. Cornea, and F. Lalonde, editors, *Morse Theoretic Methods in Nonlinear Analysis and Symplectic Topology*, volume 217 of *Nato Science Series II: Mathematics, Physics and Chemistry*, pages 1–74. Springer-Verlag, 2006.

[2] A. Abbondandolo, A. Portaluri, and M. Schwarz. The homology of path spaces and Floer homology with conormal boundary conditions. *J. Fixed Point Theory Appl.*, 4(2):263–293, 2008.

[3] A. Abbondandolo and M. Schwarz. On the Floer homology of cotangent bundles. *Comm. Pure Appl. Math.*, 59:254–316, 2006.

[4] A. Abbondandolo and M. Schwarz. Estimates and computations in Rabinowitz-Floer homology. *J. Topol. Anal.*, 1(4):307–405, 2009.

[5] A. Abbondandolo and M. Schwarz. A smooth pseudo-gradient for the Lagrangian action functional. *Adv. Nonlinear Studies*, 9:597–623, 2009.

[6] A. Abbondandolo and M. Schwarz. Floer homology of cotangent bundles and the loop product. *Geometry and Topology*, 14:1569–1722, 2010.

[7] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. *Geometry and Topology*, 14:627–718, 2010.

[8] P. Albers and U. Frauenfelder. Floer Homology for negative line bundles and Reeb chords in prequantization spaces. *J. Modern Dynamics*, 3(3):407–456, 2009.

[9] P. Albers and U. Frauenfelder. Infinitely many leaf-wise intersection points on cotangent bundles. *arXiv:0812.4426*, 2009.

[10] P. Albers and U. Frauenfelder. Leaf-wise intersections and Rabinowitz Floer homology. *J. Topol. Anal.*, 2(1):77–98, 2010.

[11] P. Albers and U. Frauenfelder. Rabinowitz Floer homology: A Survey. *arXiv:1001.4272*, 2010.

[12] P. Albers and U. Frauenfelder. A Remark on a Theorem by Ekeland-Hofer. *to appear in Isr. J. Math.*, 2010.

[13] P. Albers and U. Frauenfelder. Spectral Invariants in Rabinowitz Floer homology and Global Hamiltonian perturbations. *J. Modern Dynamics*, 4:329–357, 2010.

[14] V. I. Arnold and A. B. Givental. Symplectic Geometry. In *Dynamical Systems*, volume IV of *Encyclopedia of Mathematical Sciences*. Springer-Verlag, 1990.

[15] A. Bahri. *Critical points at infinity in some variational problems*, volume 182 of *Pitman Research Notes in Mathematics*. Longman, 1989.

[16] C. Bounya. An exact triangle for the wrapped Floer homology of the Lagrangian Rabinowitz functional. *In preparation.*

[17] K. Burns and G. P. Paternain. Anosov magnetic flows, critical values and topological entropy. *Nonlinearity*, 15:281–314, 2002.

[18] K. Cieliebak and U. Frauenfelder. A Floer homology for exact contact embeddings. *Pacific J. Math.*, 239(2):216–251, 2009.

[19] K. Cieliebak, U. Frauenfelder, and A. Oancea. Rabinowitz Floer homology and symplectic homology. *Ann. Sci. École Norm. Sup*, 43(6):957–1015, 2010.

[20] K. Cieliebak, U. Frauenfelder, and G. P. Paternain. Symplectic topology of Mañé's critical value. *Geometry and Topology*, 14:1765–1870, 2010.

[21] K. Cieliebak, V. Ginzburg, and E. Kerman. Symplectic homology and periodic orbits near symplectic submanifolds. *Comment. Math. Helv.*, 74:554–581, 2004.

[22] G. Contreras. The Palais-Smale condition on contact type energy levels for convex Lagrangian systems. *Calc. Var. Partial Differential Equations*, 27(3):321–395, 2006.

[23] G. Contreras and R. Iturriaga. *Global Minimizers of Autonomous Lagrangians*, volume 22 of *Colloqio Brasileiro de Matematica*. IMPA, Rio de Janeiro, 1999.

[24] G. Contreras, R. Iturriaga, G. P. Paternain, and M. Paternain. The Palais-Smale Condition and Mañé's Critical Values. *Ann. Henri Poincaré*, 1:655–684, 2000.

[25] J. J. Duistermaat. On the Morse Index in Variational Calculus. *Adv. Math.*, 21:173–195, 1976.

[26] U. Frauenfelder. The Arnold-Givental conjecture and moment Floer homology. *Int. Math. Res. Not.*, 42:2179–2269, 2004.

[27] V. Ginzburg. On closed trajectories of a charge in a magnetic field. An application of symplectic geometry. In C. B. Thomas, editor, *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publications of the Newton Institute*, pages 131–148. Cambridge University Press, 1996.

[28] J. Kang. Existence of leafwise intersection points in the unrestricted case. *arXiv:0910.2369*, 2009.

[29] J. Kang. Generalized Rabinowitz Floer homology and coisotropic intersections. *arxiv:1003.1009*, 2010.

[30] J. Kang. In preparation. 2011.

[31] R. Mañé. Lagrangian flows: the dynamics of globally minimizing orbits. In J. Lewowicz F. Ledrappier and S. Newhouse, editors, *International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañé)*, volume 362 of *Pitman Research Notes in Math.*, pages 120–131. Longman, 1996.

[32] L. Macarini, W. J. Merry, and G. P. Paternain. On the growth rate of leaf-wise intersections. *arXiv:1101.4812, to appear in J. Symplectic Geometry*, 2011.

[33] W. J. Merry. On the Rabinowitz Floer homology of twisted cotangent bundles. *arXiv:1002.0162, to appear in Calc. Var. Partial Differential Equations*, 2011.

[34] W. J. Merry. *Rabinowitz Floer homology and Mañé supercritical hypersurfaces.* PhD thesis, University of Cambridge, 2011. Available online at http://www.srcf.ucam.org/ wjm29/ (November 2011 onwards).

[35] W. J. Merry and G. P. Paternain. Index computations in Rabinowitz Floer homology. *J. Fixed Point Theory Appl.*, 10(1):88–111, 2011.

[36] G. P. Paternain. Magnetic Rigidity of Horocycle Flows. *Pacific J. Math.*, 225:301–323, 2006.

[37] M. Pozniak. Floer homology, Novikov rings and clean intersections. In Y. Eliashberg, D. Fuchs, T. Ratiu, and A. Weinstein, editors, *Northern California Symplectic Geometry Seminar*, volume 196 of *2*, pages 119–181. Amer. Math. Soc., 1999.

[38] P. H. Rabinowitz. Periodic solutions of Hamiltonian systems. *Comm. Pure. Appl. Math.*, 31:157–184, 1978.

[39] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32:827–844, 1993.

[40] D. Salamon. Lectures on Floer Homology. In Y. Eliashberg and L. Traynor, editors, *Symplectic Geometry and Topology*, volume 7 of *IAS/Park City Math. Series*, pages 143–225. Amer. Math. Soc., 1999.

[41] M. Schwarz. On the action spectrum for closed symplectically aspherical manifolds. *Pacific J. Math.*, 193(2):419–461, 2000.

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